Gaps in the sequence $\sqrt{n} \pmod 1$

Based on Elkies-McMullen (2004, Duke Math Journal)

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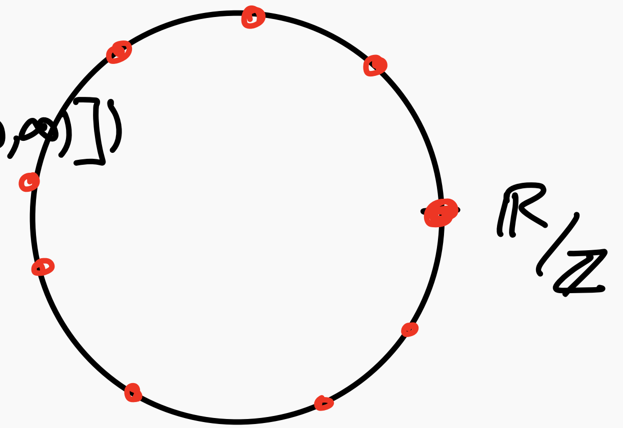
Set up

Thet set

$$\{\sqrt{k}\} : 1 \le k \le N$$

divide $S^1 = \mathbb{R}/\mathbb{Z}$ into N intervals $J_1, \ldots, J_N$. Set $a_i := N|J_i|$ for $1 \le i \le N$ and

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{a_i}. \in P([0,\infty])$$



Quesntion: Does there exists a limiting distribution λ of μ_N as $N \to \infty$. I.e. for all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$

$$\int_{\mathbb{R}} \underbrace{f(x)\, \overset{d\mu_N(x)}{\cancel{dx}}}_{= \frac{1}{N}\sum_{i=1}^{N} f(a_i)} \to \int_{\mathbb{R}} f(x) d\lambda(x).$$

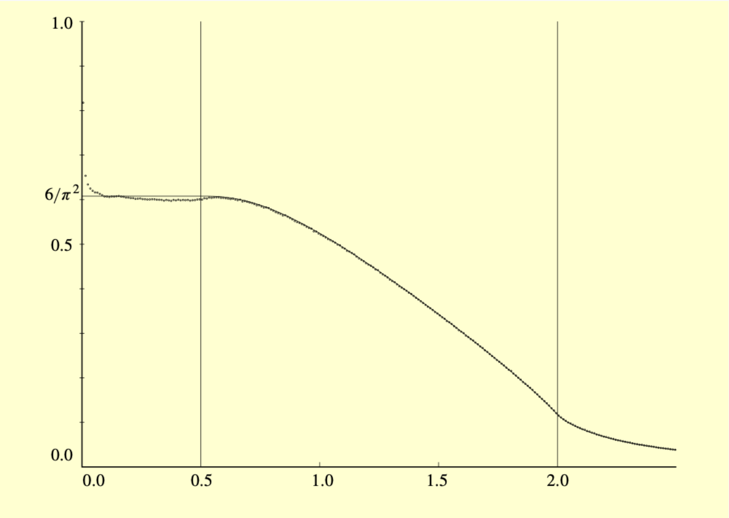
Theorem

Theorem (Elkies-McMullen, 2004)

As $N \to \infty$, μ_N converges weakly to a probability measure continuous with respect to the Lebesgue whose density function is given by

$$F(t) = \begin{cases} 6/\pi^2 & \text{if } 0 \leq t \leq 1/2 \\ F_2(t) & \text{if } 1/2 \leq t \leq 2 \\ F_3(t) & \text{if } t \geq 2 \end{cases}$$

where $F_2(t)$ and $F_3(t)$ are explicit analytic functions.



Goal of today's lecture

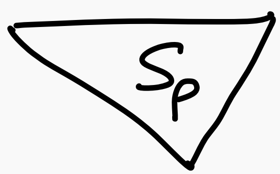
Theorem (Elkies-McMullen)

The probability $p(\rho)$ that a random unimodular affine lattice Λ in $\mathbb{R}^2$ intersects a given triangle S_ρ of area ρ satisfies

$$p'' = -F,$$

where F is as above.

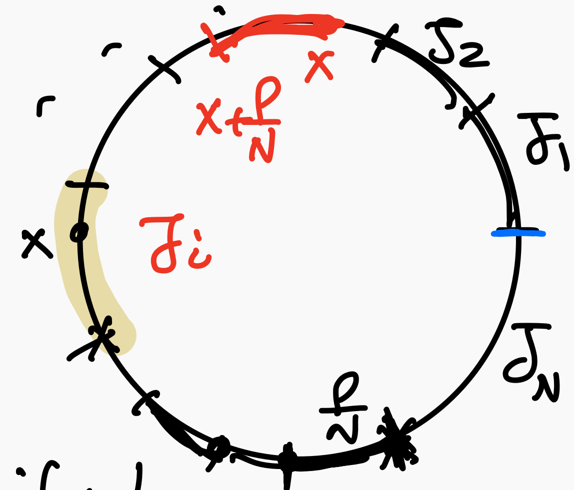
$$p(\rho) = \mu_E \{\Lambda \in \mathcal{L}^{aff}(\mathbb{R}^2) : \Lambda \cap S_\rho \neq \emptyset\}$$



interested in

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{a_i}$$

$$a_i = N|J_i|$$



Fix $\rho>0$. Pick $x \in \mathbb{R}/\mathbb{Z}$ random & uniformly.

$$I_x = [x, x+\tfrac{\rho}{N}], \quad \mathbb{P}[\, I_x \cap R_N \neq \emptyset \,]$$

$$R_N = \{ \{\sqrt{k}\} : 1 \le k \le N \}$$

$$P(\rho) = \mathbb{P}(I_x \cap R_N \neq \emptyset) = \sum_{|J_i| < \frac{\rho}{N}} |J_i| + \sum_{|J_i| > \frac{\rho}{N}} \frac{\rho}{N}$$

$$P(\rho+\delta) = \; . \; - \; - \; = \sum_{|I_i| < \frac{\rho+\delta}{N}} + \sum_{|J_i| > \frac{\rho+\delta}{N}} \frac{\rho+\delta}{N}$$

$$\frac{P_N(\rho+\delta)-P(\rho)}{\delta} = \sum_{\underbrace{|y_i|}>\frac{\rho}{N}} \frac{1}{N} = \underline{\underline{\frac{1}{N}}}\,\#\{\, i \mid \underline{\underline{a_i}} > \tfrac{\rho}{N} \}$$

$$= \mu_N([\rho,\infty))$$

$\delta \to 0,$

$$\to P_N'(\rho) = \mu_N(\rho) = 1 - \underbrace{\mu_N((-\infty,\rho])}_{N\to\infty}$$

$$\underline{N\to\infty} \quad \boxed{P_N''(\rho) = -F_N(\rho)}$$

$$\{\sqrt{n}\} \in \underbrace{[x,\, x+\tfrac{\rho}{N}]} = I_x$$

$$\Longleftrightarrow \quad \sqrt{n} \in a + I_x = [a+x,\ a+x+\tfrac{\rho}{N}]$$

$$1 \le n \le N \qquad n \in [a+x,\ a+x+\tfrac{\rho}{N}]^2$$

$$a = [\sqrt{n}] \qquad = [(a+x)^2,\ (a+x+\tfrac{\rho}{N})^2]$$

$$= [a^2+x^2+2ax,\ a^2+x^2+2ax+2a\tfrac{\rho}{N}+2x\tfrac{\rho}{N}+(\tfrac{\rho}{N})^2]$$

$$= a^2 - x^2 + [2x^2+2ax,\ 2x^2+2ax+2a\tfrac{\rho}{N}+2x\tfrac{\rho}{N}]$$

$$y = a^2 - x^2 + 2(a+x)\underbrace{\left[x, x+\frac{\rho}{N}\right]}_{I}$$

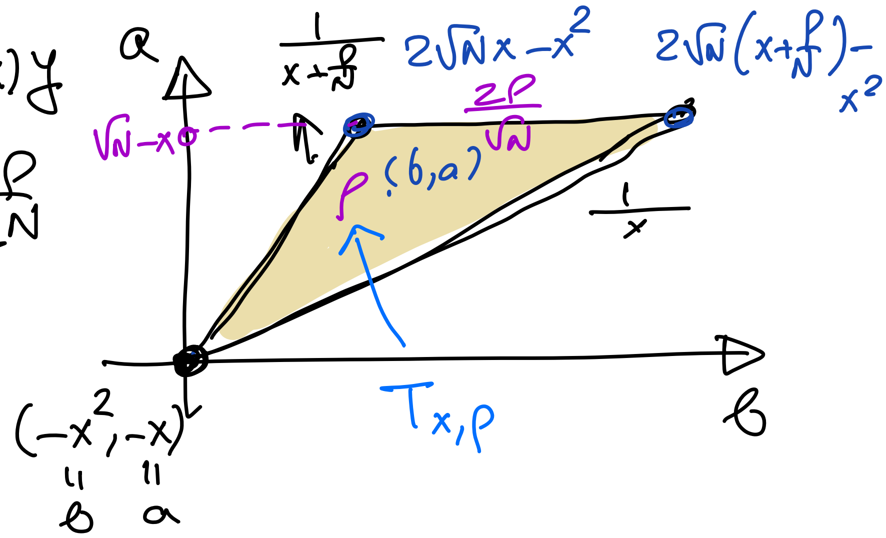
$$\underbrace{b - a^2}_{b\in\mathbb{Z}} + x^2 = 2(a+x)y \qquad y \in I$$

$a \in \mathbb{Z}$ $\qquad 0 \le a + x \le N$

$$b + x^2 = 2(a+x)y$$

$$x \le y \le x + \frac{\rho}{N}$$

$x \in [0,1]$

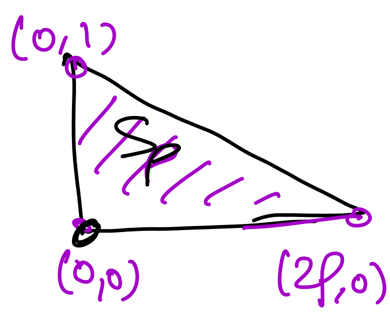


$$\mathbb{P}[I_x \cap R_N \neq \emptyset] = \mathbb{P}_{x\in[0,1]}[\mathbb{Z}^2 \cap T_{x,\rho} \neq \emptyset]$$

find $g_{x,N}$

$$g_{N,x} T_{x,\rho} = S_\rho$$

$$g_{N,x}\begin{pmatrix}-x^2\\ -x\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$



affine lattice

$$= \mathbb{P}[\underbrace{g_{N,x}\mathbb{Z}^2}_{0\le x\le 1} \cap S_\rho = \emptyset] \xrightarrow[N\to\infty]{} \mathbb{P}[\{\Lambda : \Lambda \cap S_\rho \neq \emptyset\}]$$

with res μ_E

$$g_{x,N} = \begin{pmatrix} \sqrt{N} & -2x\sqrt{N} & -x^2\sqrt{N} \\ 0 & 1/\sqrt{N} & x/\sqrt{N} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{N} & -2x/\sqrt{N} \\ 0 & 1/\sqrt{N} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x^2/\sqrt{N} \\ x/\sqrt{N} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{N} & 0 & 0 \\ 0 & \frac{1}{\sqrt{N}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2x & -x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

$$\| $$

$$g_x \quad 0 \le x \le 1$$

$$g_{x+1} = g_x \cdot \gamma \quad \gamma \in SL_2\mathbb{Z} \ltimes \mathbb{Z}^2 = \Gamma$$

Space of affine lattices

Groups

1.
$$G = \left\{ g = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} : ad - bc = 1 \right\} \subseteq SL_3 \mathbb{R}$$

" $Aff(\mathbb{R}^2)$ "

2. G acts on $\mathbb{R}^2$ via
$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}.$$

3.
$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad V = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq \mathbb{R}^2$$

4. $1 \to V \to G \xrightarrow{D} \mathrm{SL}_2(\mathbb{R}) \to 1$

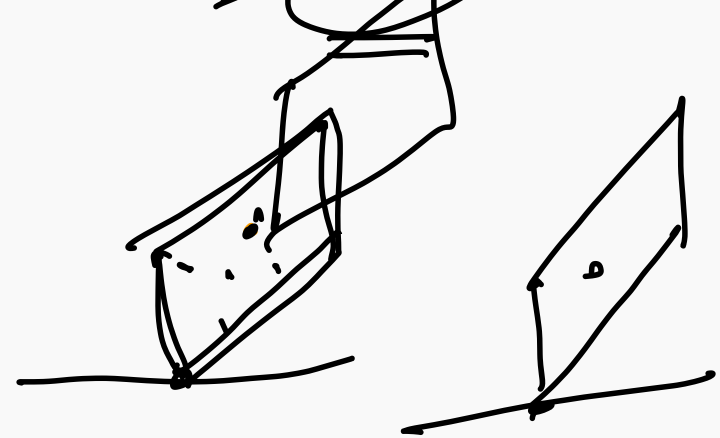
groups and spaces

- $\Gamma = G \cap \mathrm{SL}_3(\mathbb{Z})$
- $\mathcal{L} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$, equipped with the probability measure μ_B
- $\mathcal{L}^{aff} = G/\Gamma \simeq \mathrm{ASL}_2(\mathbb{R})/\mathrm{ASL}_2(\mathbb{Z})$, equipped with the probability measure μ_E
- $$D : \mathrm{ASL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$$

  defined by

  $$Ax + b \to A$$

$D : \mathcal{L}^{aff} \to \mathcal{L}$



- $\mathcal{L}^{aff}$ is a torus bundle over $\mathcal{L}$.

- $A = \left\{ \underbrace{A_s := \begin{pmatrix} s & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{pmatrix}} : s \neq 0 \right\}$, $s \in \mathbb{R}$ $\qquad N = \left\{ \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.
- Geodesic flow on $\mathcal{L}$ defined by g_s
- Teichmüller flow on $\mathcal{L}^{aff}$ defined by $\{A_s : s > 0\}$. $\qquad \mathcal{L}^{aff}$
- The Teichmüller flow is ergodic and mixing.

$D: \mathcal{L}^{aff} \longrightarrow \mathcal{L}$

$A_s \qquad \left\{ g_s = \begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix} \right\} \subset SL_2\mathbb{R}$

V_2

$\{ g_x \mathbb{Z}^2 : 0 \leq x \leq 1 \} \subseteq \mathcal{L}^{aff}$

loop

we consider measures

$$m_N = \text{Push forward of the Lebesgue measure on } [0,1] \text{ with respect to } A_s \cdot \sigma$$

$$A_s = \begin{pmatrix} s & 0 & 0 \\ 0 & s^{-1} & 0 \\ 0 & & 1 \end{pmatrix} \qquad A_{\sqrt{N}} = \begin{pmatrix} \sqrt{N} & 0 & 0 \\ 0 & \frac{1}{\sqrt{N}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma(t) = \begin{pmatrix} 1 & -2t & -t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma : [0,1] \longrightarrow G.$$

- After passing to a subsequence we can assume

$$\sigma_{N_i} \longrightarrow \underline{\underline{\mu}} \quad (w^*)$$

$$\boxed{D_* \mu = \mu_B} = \mu_{SL_2\mathbb{R}/SL_2\mathbb{Z}}.$$

- There is no escape of mass.

N-invariantce

More generally, we consider

$$N_\tau = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma_N(t) = \begin{pmatrix} \sqrt{N} & \sqrt{N}t & \sqrt{N}x(t) \\ 0 & 1/\sqrt{N} & y(t)/\sqrt{N} \\ 0 & 0 & 1 \end{pmatrix}$$

$$N_\tau \sigma_N(t) = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sigma_N(t)$$

$$y_N(t) = \begin{pmatrix} \sqrt{N} & \sqrt{N}\left(t + \frac{\tau}{N}\right) & \sqrt{N}\, x(t) + \frac{1}{\sqrt{N}} \tau y(t) \\ 0 & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} y(t) \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_N(t) = \eta_N\left(t - \frac{\tau}{N}\right) = \begin{pmatrix} \sqrt{N} & \sqrt{N}\,t & \sqrt{N}\,x\left(t-\frac{\tau}{N}\right) + \frac{1}{\sqrt{N}}\,y\left(t-\frac{\tau}{N}\right) \\ 0 & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}}\,y\left(t-\frac{\tau}{N}\right) \\ 0 & 0 & 0 \end{pmatrix}$$

$\underline{\underline{t \in [0,1)}}$

$$m(P_N) \xrightarrow{w^*} (N_\tau)_* \mu \qquad \mu = \lim_{w^*} (\sigma_N)$$

$$O\left(\frac{\tau}{N^2}\right)$$

$$\underline{\underline{\sqrt{N}}}\left(x\left(t-\frac{\tau}{N}\right) - x(\tau)\right) +$$

$$\frac{1}{\sqrt{N}}\left|y\left(t-\frac{\tau}{N}\right) - y(t)\right| + \dots$$

__So__ ~~m~~ limiting meaure is N-invariant.

Ratner's theorem

Theorem (Ratner)

Let Γ be a discrete subgroup of a Lie group G and N be a unipotent subgroup of G. Let ν be an ergodic N-invariant probability measure on G/Γ and $J = stab(\nu)$. Then there exists $x \in G/\Gamma$ such that $\nu(J \cdot x) = 1$. Moreover, $J \cdot x$ is closed in G/Γ and coincides with the support of ν.

μ is N-invariant

$$D_* \mu = \mu_B$$

- study possible candidates for ergodic measures.

ergodic decomposition

$$\mu = \int_X \nu \, dP(\nu)$$

Prob. on space of N-invariant

$$\underbrace{\mu_B}_{\text{}} = D_* \mu = \int \underbrace{D_* \mu}_{} \, dP(\nu)$$

ergodic with respect to $\underline{N}$-action

- ν ergodic
- $D_* \nu = \mu_B$
- ν N-invariant.

Let $J = \mathrm{stab}(\nu)$

Ratner: support $\nu = J.x$ for some $x \in G/\Gamma$.

apply D:

$D(J) D(x) = D(J.x) = D(\text{Support } \nu) =$

$\text{Support } D_* \nu = \underline{SL_2\mathbb{R}/SL_2\mathbb{Z}}$.

$\boxed{D(J) = SL_2(\mathbb{R}).}$

$D : ASL_2 \mathbb{R} \to SL_2(\mathbb{R})$

$\quad \parallel$

$SL_2 \mathbb{R} \ltimes \mathbb{R}^2$

$\mathrm{Ker}(J \to SL_2 \mathbb{R}) \subseteq \mathbb{R}^2$ — $\underline{\mathbb{R}^2}$; $\{0\} \to \underline{G}$

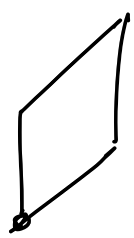
$E : SL_2 \mathbb{R} \longrightarrow SL_2 \mathbb{R} \ltimes \mathbb{R}^2$ $\quad \mathrm{stab}\begin{pmatrix} a \\ b \end{pmatrix}$ $\quad \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$

Possible candidates for H

- $H = G$
- $H = g\, SL_2(\mathbb{R})\, g^{-1}$ $\quad g = \left(\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$

<u>Orbit classification</u>

$\underline{J = SL_2\mathbb{R}} \curvearrowright ASL_2\mathbb{R}/ASL_2\mathbb{Z}$



$SL_2\mathbb{R} \ltimes \mathbb{R}^2$

E affine lattices (Λ, v) $\quad v \in \mathbb{R}^2/\Lambda$

Fix n

$E_n = \{(\Lambda, v) : nv \in \Lambda\} \subset E$

every closed orbit $\subset E_n$ for <u>some n</u>.

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in the ergodic decomposition of μ we

can have μ_E as

measures μ $\quad \operatorname{supp} \mu \subseteq gE[n] \qquad g = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\sigma_N(t) \in \boxed{\cancel{H}(r,\varepsilon)\, E[n]} \qquad \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & \circledast \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} * \to |\ | < r \\ \circledast \to \varepsilon \end{matrix}$$

$\uparrow$ Thickend set

$$\sigma_N(t) = \begin{pmatrix} \sqrt{N} & -2t\sqrt{N} & -t^2\sqrt{N} \\ 0 & \frac{1}{\sqrt{N}} & \frac{t}{\sqrt{N}} \\ 0 & 0 & 1 \end{pmatrix}$$

Elements in $E[n]$ have the form

$$\begin{pmatrix} a & b & \frac{ai+bj}{n} \\ c & d & \frac{ci+dj}{n} \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with } i,j \in \mathbb{Z}$$

$X_{ij}^N \qquad \left| -t^2\sqrt{N} - \frac{i}{n}\sqrt{N} - \frac{j}{n}t\sqrt{N} \right| < r$

$X_{ij}^N \qquad \boxed{\left| t^2 + \frac{j}{n}t + \frac{i}{n} \right| < \frac{r}{\sqrt{N}}} \qquad \frac{1}{\sqrt{N}\,j}$

$Y_{ij}^N : \quad \left| \frac{t}{\sqrt{N}} - \frac{j}{n} \right| < \varepsilon$

$Y_N^{ij} \qquad \left| t - \frac{j}{n} \right| < \varepsilon\sqrt{N}$

- fix i,j $\quad m(X_{ij}^N) \to 0$ as $N \to \infty$

$$|j| > 10n \Rightarrow m(X_{ij}^N) < \frac{r}{\sqrt{N}}$$

$$Y_{ij} \neq \emptyset \Longrightarrow \left|\frac{j}{n} - t\right| < \varepsilon\sqrt{N}$$

$$\text{if } N > 1/\varepsilon \Rightarrow \left|\frac{j}{n}\right| < 1 + \varepsilon\sqrt{N} < 2\varepsilon\sqrt{N}$$

$$\left|\frac{i}{n}\right| < \frac{r}{\sqrt{N}} + \frac{j}{n}t + t^2$$

$$\leq 1 + j + 1 = j + 2$$

bound on i $\quad \boxed{i < n(j+2)}$

$$m(T_N) = \sum_{j < 2n\varepsilon\sqrt{N}} \sum_{|i| < n(j+2)} m(X_{ij}^N)$$

$$\leq \sum_{10n < j < 2n\varepsilon\sqrt{N}} \underbrace{\sum_{|i| < n(j+2)}} m(-)$$

$$\leq \frac{1}{\sqrt{N}\, j}\, j \sqrt{N} \varepsilon = O(\varepsilon)$$

tends to 0 as $\varepsilon \to 0$.