# Sofic groups are surjunctive

Keivan Mallahi-Karai

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Constructor University

Based on the paper *Sofic groups and dynamical systems* by Benjamin Weiss.

# Dynamics on shift spaces

- $G$ finitely generated group $\quad \Omega = \{f : G \to A\} \quad G = \mathbb{Z}$
- $A = \{1, \dots, a\}$, where $a \geq 2$.
- $\Omega = A^G$ equipped with the product topology: compact and Hausdorff
- $G$ acts on $\Omega$ via

$$\sigma_g(\omega)(h) = \omega(hg) \in \Omega$$

- Connection between group theoretical properties of $G$ and dynamical properties of this action?
- (E. Glasner, B.Weiss, 1997) $G$ has Property (T) iff the set of extreme points of the simplex of invariant measures is closed.

# Main result of this presentation

**Definition**
Let $X$ be a compact metric space equipped with a continuous $G$-action. We say that $(G, X)$ is surjunctive if for every continuous $\phi : X \to X$ satisfying $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$, if $\phi$ is injective then it is surjective.

$G$ is surjunctive if the shift space action $(A^G, G)$ is surjunctive for all $a$.

# Question

Let $P$ be a finite set with at least two elements, let $P$ be provided with its discrete topology, let $T$ be an infinite discrete group, let $X$ be the cartesian power $P^T$ provided with its product topology, and let $T$ act upon $X$ by left translation. Then $(X, T)$ is called the *left symbolic transformation group over* $T$ *to* $P$. If $T$ is the additive group $\mathbb{Z}$ of integers, then $(X, T)$ is the standard symbolic flow. In general, $X$ is compact Hausdorff zero-dimensional self-dense, and $(X, T)$ is expansive. A presumably large project is to correlate group properties of $T$ with dynamical properties of $(X, T)$. Here are some recent results of Wayne Lawton [5, 6] in this context:

(1) $T$ is profinite iff the set of periodic points of $(X, T)$ is dense in $X$.

(2) Call $T$ *surjunctive* in case every one-to-one endomorphism of $(X, T)$ is onto for all $P$. If $T$ is locally finite or profinite or abelian, then $T$ is surjunctive. Also every subgroup of a surjunctive group is surjunctive.

No example of a non-surjunctive group seems to be known. If it could be proved that every quotient group of a surjunctive group is surjunctive, then every group would be surjunctive.

Professor Hedlund has pointed out that every symbolic flow is surjunctive [3].

source: Walter Gottschalk, *Some general dynamical notions*. Recent advances in topological dynamics, proceedings of the conference held at Yale University, June 19-23 1972.

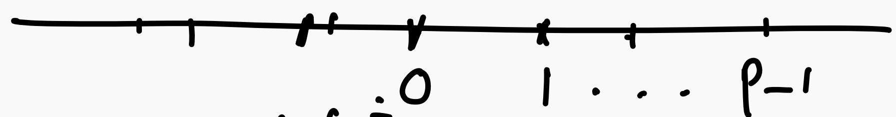
# $\mathbb{Z}$ is surjunctive: Proof I

$A = \{1, \dots, a\}$

$\varphi : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ continuos, $\mathbb{Z}$-~~invariant~~ equivariant

$\varphi$ injective $\Longrightarrow$ $\varphi$ surjective

$p \geqslant 1$ $\quad X_p = \{\omega \in A^{\mathbb{Z}} : \omega(n+p) = \omega(n) \ \forall n \in \mathbb{Z}\}$

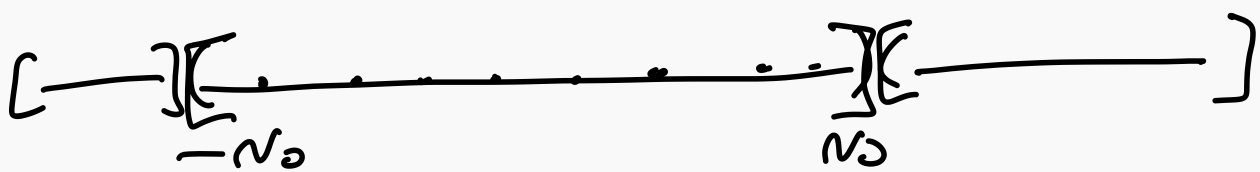



$|X_p| = a^p$

$\varphi(X_p) \subseteq X_p$ $\qquad \varphi$ injective $\xRightarrow{X_p \text{ finite}}$ $\varphi$ is surjective

$\mathrm{Im}(\varphi) \supseteq X_p \quad \forall p$

- $\overline{\bigcup_{p \geqslant 1} X_p} = A^{\mathbb{Z}}$




$\mathrm{Im}\, \varphi = A^{\mathbb{Z}}.$

**Definition**
A group $G$ is called residually finite if for every finite set $S \subseteq G$, there exists a finite quotient $\pi : G \to F$ of $G$ such that $\pi|_S$ is injective.

E.g. $\mathbb{Z}$ $\quad S \subset \mathbb{Z}$ finite $\quad N$ large integer $\quad \mathbb{Z} \xrightarrow{\pi_N} \mathbb{Z}/N\mathbb{Z}$

$\pi_N|_S$ is injective $\Longrightarrow$ $\mathbb{Z}$ is residually finite.

$G$ residually finite

$N$ finite index subgroup of $G$ $\quad F = G/N \quad G \xrightarrow{\pi_N} G/N$

$$X_N = \{\omega \in A^G \mid \omega(gn) = \omega(g) \ \forall n \in N\}$$

$|X_N| = |A|^{|F|} \qquad \varphi(X_N) \subset X_N$ injective $\Rightarrow \varphi(X_N) = X_N$

$\operatorname{Im} \varphi \supseteq \bigcup_{\substack{N \trianglelefteq G \\ \text{finite index}}} X_N \qquad S \subseteq G \quad S = \{g_1, \dots, g_k\}$

$\omega(g_1) = a_1, \dots, \omega(g_k) = a_k.$

# $\mathbb{Z}$ is surjunctive: Proof II

**Definition**
For a closed $\cancel{G}\,\mathbb{Z}$-invariant subset $X \subseteq A^{\cancel{G}\,\mathbb{Z}}$, define its entropy via

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |X_n| = \inf_{n \geq 1} \frac{1}{n} \log |X_n|.$$

where $X_n \subseteq A^{\{0,\dots,n-1\}}$ is the image of $X$ via

$$\pi(x_k)_{k\in\mathbb{Z}} = (x_k)_{0\leq k\leq n-1}.$$

The convergence follows from $|X_{m+n}| \leq |X_m||X_n|$ and Fekete's lemma.

$X$ closed invariant set

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |X_n| \leq \log a.$$

If $X \neq A^G$ then $h(X) < \log a$.

$X \neq A^G \Rightarrow \exists n$ s.t.

$$X_n \neq A^{\{0,\dots,n-1\}}.$$

$|X_m| < a^m - 1$ for some $m$

$$h(X) = \inf_n \frac{1}{n} \log |X_n| \leq \frac{1}{m} \log (a^m - 1) < \log a.$$

$\varphi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ continuous + $\mathbb{Z}$-invariant
injective

$X = \varphi(\underline{\underline{A^{\mathbb{Z}}}}) \subset A^{\mathbb{Z}}$

- $X$ compact, $\quad \psi : \underline{\underline{X}} \to A^{\mathbb{Z}}$ bijection
continuous.

$\omega \in X$ and $\quad \psi(\omega)(0)$

$\exists N$ such that $\psi(\omega)(0)$ is determined by

$\underline{\underline{\omega|_{[-N,N]}}}$ so $\quad \psi\underline{\underline{(\omega)}}(0) = \psi_{loc}(\omega_{[-N,N]})$.

$\mathbb{Z}$-invariance $\quad \underline{\psi(\omega)(n)} = \psi_{loc}(\underline{\omega|_{[-N+n, N+n]}})$

$\psi(\omega)(n) = (\sigma_{\underline{\underline{n}}}\,\omega)(0)$

$[0, n)$

$X = \underbrace{\varphi(A^{\mathbb{Z}})}$ $\quad \pi_n : A^{\mathbb{Z}} \longrightarrow A^{\{0,\dots,n+2N\}}$

$$|\pi_n(X)| \geq |\underline{\underline{A}}|^n.$$

c

$$\omega = ( \boxed{\underset{a_0}{\omega(0)}, \underset{a_1}{\omega(1)}, \dots, \underset{a_{n-1}}{\omega(n-1)},} \dots ) \xrightarrow{\varphi} \underline{\qquad} \quad \varphi(\omega) \in \underline{\underline{X}}$$

$$|\pi_n(X)| \geq a^n$$

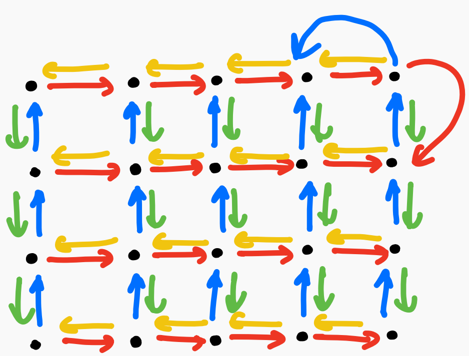
$$h(\cancel{\pi} X) = \lim_{n\to\infty} \frac{1}{n+2N} \log a^n = \underline{\underline{\log a}}$$

# Cayley graphs of finitely generated groups

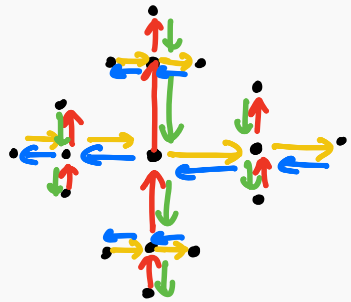
**Definition**
Let $G$ be a group generated by a finite symmetric set $S$. The Cayley graph of $G$ with respect to $S$, denoted by $\mathrm{Cay}(G, S)$ is the graph with the vertex set $G$, with a directed edge from $g$ to $sg$ for each $g \in G$ and $s \in S$ labeled with $s$.

- The graph metric on $G$ is denoted by $d$.
- Ball of radius $r$ centered at 1 is denoted by $B_S(r)$.

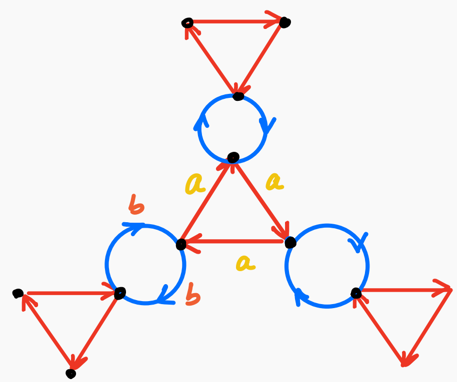
$G = \mathbb{Z} \times \mathbb{Z}$



$G = F_2$



$G = \langle a, b \mid a^3 = b^2 = 1 \rangle$



# Sofic groups

**Definition**
A finitely generated group $G$ is sofic if for some finite symmetric finite set $S$ and every $\epsilon > 0$ and $r \geq 1$ there exists a finite graph with the vertex set $V$ with (directed) edges labeled by $S$ with a subset $V_0 \subseteq V$ such that

- For every $v \in V_0$ the $r$-ball centered at $v$ in $V$ is isomorphic (as a labeled graph) to $B_S(r)$.
- $|V_0| \geq (1 - \epsilon)|V|$.

# Examples of sofic groups

Ex 1: Residually finite groups

Ex 2: amenable groups

# Sofic groups are surjunctive

**Theorem (Gromov 1999, Weiss 2000)**

*Every sofic group is surjunctive.*

$$\varphi : A^G \longrightarrow A^G \qquad X = \varphi(A^G)$$

$$\psi : X \twoheadrightarrow A^G \qquad \psi \circ \varphi = \mathrm{id}_{A^G}.$$

$$\exists\ \varphi_{loc} : A^{B(r_0)} \longrightarrow A \qquad \varphi(\omega)(e) = \varphi_{loc}(\omega|_{B(r_0)}).$$

$$\exists\ \psi_{loc} : X_{r_0} \subset A^{B(r_0)} \longrightarrow A \qquad \psi(\omega)(e) = \psi_{loc}(\omega|_{B(r_0)}).$$

$n \geq 1$

~~Rat~~ Notation: $V(nr_0) = \{ v \in V \mid B_{nr_0}(v) \cong B(nr_0) \}$

~~sketch of the~~ proof:

- ~~Local functions and their~~ properties
- 

$$\varphi_n : A^{V(nr_0)} \longrightarrow A^{V((n+1)r_0)}.$$

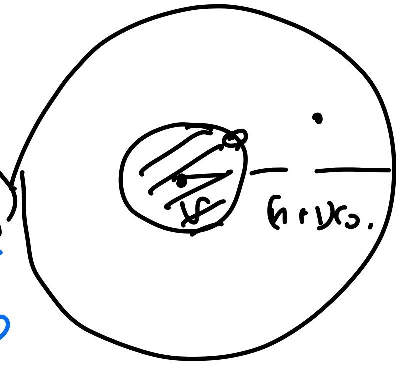
$\omega \in A^{V(nr_0)}$

we want to define

$$v \in V((n+1)r_0). \quad \varphi_n(\omega)(v) = \varphi_{loc}\Big(\omega|_{\underset{\underset{B(r_0)}{\cong}}{B(v,r_0)}}\Big)$$

$$\Downarrow$$

$$B_{r_0}(v) \subseteq V(nr_0).$$




$\psi_n$ defined in a similar way using $\psi_{loc}$.

$$A^{V(nr_0)} \xrightarrow{\varphi_n} A^{V((n+1)r_0)} \xrightarrow{\psi_{n+1}} A^{V((n+2)r_0)}.$$

$$\psi \circ \varphi = id \Longrightarrow \quad \psi_{n+1} \circ \varphi_n(\underset{A^{V(nr_0)}}{\underset{\uparrow}{\omega}}) = \omega|_{\underset{V(nr_0).}{\underset{\cap}{V((n+2)r_0)}}}$$

$n = 1$

<u>Upper bound</u>

$$Z = \varphi_1(A^{V(r_0)}) \subseteq A^{V(2r_0)}.$$

$$\psi_2(Z) = \psi_2(\varphi_1(A^{V(r_0)})) = A^{V(3r_0)}$$

$$\Longrightarrow \quad |Z| \geq |\psi_2(Z)| = |A|^{|V(3r_0)|}.$$

<u>Lower bound.</u>

<u>Claim</u> $\exists$ subset $U \subseteq V(3r_0)$ with the properties:

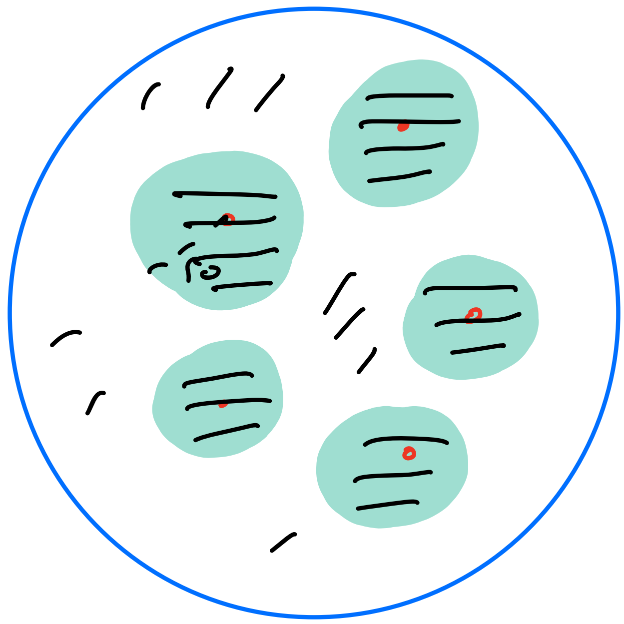
- $\{B(u, r_0)\}_{u \in U}$ are pairwise disjoint
- $$|U| \geq \frac{|V(3r_0)|}{|B(2r_0)|}$$

$V(3r_0) \subset V$

$$Z = \varphi_1(A^{V(r_0)})$$

Suppose that $\varphi(A^G) \neq A^G$.

$\varphi(A^G)$ is a proper closed subset of $A^G$.




$\exists r_0$ such that

$$\pi_{B(r_0)}(\varphi(A^G)) \subsetneq A^{B(r_0)}.$$

$$|\pi_{B(r_0)}(\varphi(A^G))| \leq |A|^{|B(r_0)|} - 1.$$

$$|A|^{|V(3r_0)|} \leq |Z| \leq (|A|^{B(r_0)} - 1)^{|U|} \cdot |A|^{|V(2r_0)| - |U| \cdot |B(r_0)|}$$

$|A| = a$

$$\Rightarrow \quad a^{|V(3r_0)|} \leq \left(1 - \frac{1}{a^{|B(r_0)|}}\right)^{|U|} \cdot a^{|V|}$$

$$a^{\frac{|V(3r_0)| - |V|}{|V(3r_0)|}} \leq \left(1 - \frac{1}{|a|^{|B(r_0)|}}\right)^{\frac{|U|}{|B(3r_0)|}}$$

$$a^{1-\frac{|V|}{|V(B(r_0))|}} \le \left(1 - \frac{1}{|a|^{|B(r_0)|}}\right)^{\frac{1}{|B(2r_0)|}}$$

$a^{1-(1+\varepsilon)} \le C$ fixed only depends on $r_0$

$a^{\varepsilon} \ge C \quad \forall \varepsilon$

Thank you!