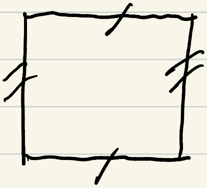
 $= \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}$

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T} \circlearrowleft$

It has 2 eigenvalues $\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$

has corresponding e-directions $\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix}$

$\mathbb{T}$ has a straight line flow in every direction, these flows are isometries & preserve Lebesgue measure.

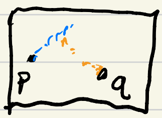
Let F^t = flow in direction $\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ on $\mathbb{T}$

Goal: Show this flow is (uniquely) ergodic.

We will do this by showing if $f \in Lip_1(\mathbb{T})$ & $x, y \in \mathbb{T}$ then

$$\lim_{N\to\infty} \frac{1}{N}\int_0^N f(F^t x)\,dt = \lim_{N\to\infty} \frac{1}{N}\int_0^N f(F^t y)\,dt.$$

Fact: If p, q are in $\mathbb{T}$ then $\exists\ \vec{u}$ & $\vec{v}$ w/ $|\vec{u}|, |\vec{v}| \leq \sqrt{2}$ in directions $\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix}$ so that $p + \vec{u} + \vec{v} = q$.



$\mathbb{T}$

Choose constants: $\epsilon > 0$ is given

Choose n s.t. $\left(\frac{3-\sqrt{5}}{2}\right)^n \sqrt{2} < \frac{\epsilon}{2}$

this also means $\left(\frac{3+\sqrt{5}}{2}\right)^{-n} \sqrt{2} < \frac{\epsilon}{2}$.

Choose $L > \frac{2}{\epsilon} \left(\frac{3+\sqrt{5}}{2}\right)^n \sqrt{2}$.

Let x, y be any pts in $\mathbb{T}$

Prop There is $|s| < \left(\frac{3+\sqrt{5}}{2}\right)^n \sqrt{2} < \frac{\epsilon}{2} L$ so that for all t

$$d(F^t y, F^{t-s} x) < \left(\frac{3-\sqrt{5}}{2}\right)^n \sqrt{2} < \frac{\epsilon}{2}$$

picture to have in mind is $t = 0$ on s

Let $p = A^{-n}x$ & $q = A^{-n}y$

Apply the fact to p & q to get

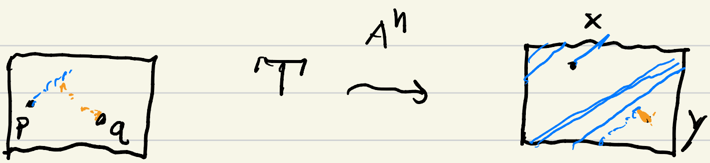
$$q = \underbrace{p}_{A^{-n}x} + \vec{u} + \vec{v} \quad w/ \ |\vec{u}|, |\vec{v}| < \sqrt{2}.$$

$y = A^n q$ & so

$$y = A^n q = A^n(A^{-n}x + \vec{u} + \vec{v}).$$

$$= x + \left(\frac{3+\sqrt{5}}{2}\right)^n \vec{u} + \left(\frac{3-\sqrt{5}}{2}\right)^n \vec{v}.$$

$$d\left(F^{\pm\left(\frac{3+\sqrt{5}}{2}\right)^n |\vec{u}|} x, y\right) = \left(\frac{3-\sqrt{5}}{2}\right)^n |\vec{V}|$$



B/C F^t is an isometry we have $\forall t$

$$d\left(F^{t \overbrace{\pm\left(\frac{3+\sqrt{5}}{2}\right)^n |\vec{u}|}^{\le \epsilon/2}} x, F^t y\right) = \left(\frac{3-\sqrt{5}}{2}\right)^n |\vec{V}| < \frac{\epsilon}{2} \quad \square$$

pf of main result

$$\left| \int_0^N f(F^t x) - \int_0^N f(F^t y) \right| < \epsilon N$$

$\forall N > L$ & $f \in Lip_1(T)$.

$$\star \int_0^N f(F^t x)\,dt - \int_0^N f(F^t y)\,dt =$$

Assume s

$$\int_0^s f(F^t x)\,dt + \int_s^N f(F^t x)\,dt - \int_0^{N-s} f(F^t y)\,dt -$$

$$\int_{N-s}^N f(F^t y)\,dt.$$

We will estimate trivially on

For it is $\int_0^{N-s} \left(f(F^{t+s} x) - f(F^t y) \right) dt$

b/c f is 1-Lip this is at most

$$(N-s)\, d(F^{t+s} x, F^t y) < \frac{\epsilon}{2}(N)$$

From we get $|s| \cdot 1\, \mathrm{diam}(T)$

$$\sqrt{2} \cdot \frac{\epsilon}{2} L + \frac{\epsilon}{2} N < \left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right) \epsilon N.$$

Idea of pf:

0) Effectivising a limiting statement

1) Contraction coming from A transverse to the dynamics (the F^t dynamics)

2) Expansion in the direction of dynamics can be treated by averaging for longer.