Quadratic twists of modular L-functions

Xiannan Li

PIMS seminar

Quadratic twists

Let f be a primitive modular form of weight κ and level N and suppose that f is a Hecke eigenform. The L-function associated with f is given by

$$L(s,f) = \sum_n \frac{\lambda_f(n)}{n^s},$$

for Re $s > 1$, and can be analytically continued to the entire complex plane.

Quadratic twists

Let d be a fundamental discriminant, and $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$ denote the primitive quadratic character with conductor $|d|$. Then $f \otimes \chi_d$ is a Hecke eigenform, with L-function given by

$$L(s, f \otimes \chi_d) = \sum_n \frac{\lambda_f(n)\chi_d(n)}{n^s} \qquad (1)$$

for Re $s > 1$.

$x^2 + y^2 + z^2 = n$

n is represented by $x^2 + y^2 + z^2$ if it is not of the form $\underbrace{4^a(8b-1)}$. When representations exist, one may ask whether the points $\frac{1}{\sqrt{n}}(x, y, z)$ equidistribute on the unit sphere.

- Linnik resolved this using ergodic methods subject to some an condition that n is a quadratic reside modulo some prime.
- Iwaniec resolved this without the assumption by bounding Fourier coefficients of half weight integer modular forms.
- By Waldspurger's formula, one may relate the Fourier coefficient of a primitive half integer weight cusp form g to L-values of a quadratic twist of a modular form f.

$$\underbrace{|\rho_g(|D|)|^2} \asymp \underbrace{L(1/2, f \otimes \chi_D)},$$

for D fundamental discriminant.

Quadratic twists of elliptic curves

We call a natural number n congruent if n occurs as the area of a right angle triangle with sides of rational length. This gives a system

$$a^2 + b^2 = c^2, \text{ and } \frac{ab}{2} = n.$$

A change of variables $x = n(a+c)/b$ and $y = 2n^2(a+c)/b^2$ gives

$$y^2 = x^3 - n^2 x,$$

and n is congruent if and only if there are solutions for the above in rational x, y with $y \neq 0$. A change of variables $(x, y) \to (nx, n^2 y)$ gives the equivalent form

$$ny^2 = x^3 - x.$$

These are the quadratic twists of

$$y^2 = x^3 - x.$$

More generally when an elliptic curve E is given by $y^2 = x^3 + ax + b$, the quadratic twist E_d is given by

$$dy^2 = x^3 + ax + b.$$

For $L(s, E)$ the L-function of E, $L(s, E_d) = L(s, E \otimes \chi_d)$ (in the sense previously introduced).
The rational points on E_d is an abelian group of finite rank r. The Birch and Swinnerton-Dyer predicts that r is the same as the order of vanishing of $L(s, E_d)$ at the critical point.

Moments

For simplicity, assume f is full level and restrict attention to fundamental discriminants of the form $8d$ where d is odd and squarefree. We let $\sum^*$ denote a sum over squarefree integers. It is of high interest to understand moments of the form

$$M(k) := \sum_{\substack{0<8d<X \\ (d,2)=1}}^{*} L(\underset{\text{critical point}}{\underbrace{1/2}}, f \otimes \chi_{8d})^k. \tag{2}$$

Moments

Keating and Snaith conjectured that

$$M(k) \sim \underbrace{C(k,f)}_{}\underbrace{X(\log X)^{\frac{k(k-1)}{2}}}_{},$$

for an explicit constant $C(k,f)$. This conjecture is analogous to conjectures for moments of other families.

- The conjecture is known for the first moment $k = 1$ by Iwaniec's work.

  $1+\varepsilon$
- Based on knowledge of the twisted first moment, Radziwill and Soundararajan proved that $M(k) \ll X(\log X)^{\frac{k(k-1)}{2}}$ for $0 \leq k \leq 1$.

The second moment

For $k = 2$,

- The work of Heath-Brown implies that $M(2) \ll X^{1+\epsilon}$.
- The method of Soundararajan gives $M(2) \ll X(\log X)^{1+\epsilon}$, and refinement by Harper gives $M(2) \ll X(\log X)$, both conditionally on GRH.
- Based on similar ideas applied to bounding shifted moments, Soundararajan and Young proved the conjectured asymptotic for $M(2)$ assuming GRH.

The second moment

Theorem

$$\sideset{}{^*}\sum_{\substack{0<8d<X \\ (d,2)=1}} L(1/2, f \otimes \chi_{8d})^2 \sim C_f \underbrace{X \log X}_{},$$

where C_f is some explicit constant depending on f.

If we include a smooth weight in the sum over d above, the result can be proven with an error term of quality $O(X\underbrace{(\log X)^{1/2+\epsilon}}_{})$ and improved to $O(X\underbrace{(\log X)^{\epsilon}}_{})$ with a little effort.

Moments of quadratic Dirichlet L-functions

Let

$$\mathcal{M}(k) = \sideset{}{^*}\sum_{\substack{0<8d<X \\ (d,2)=1}} L(1/2, \chi_{8d})^k.$$

- The first and second moments were computed by Jutila, and the third moment by Soundararajan.
- Refinements with improved error terms: on the first and third moments by Young, the second moment by Sono, and a further refinement of the third moment by Diaconu and Whitehead explicating a power saving secondary term.
- The fourth moment was computed recently assuming GRH by Shen, following the approach of Soundararajan and Young.

Moments of quadratic Dirichlet L-functions

Let

$$\mathcal{M}(k) = \sideset{}{^*}\sum_{\substack{0<8d<X \\ (d,2)=1}} L(1/2, \chi_{8d})^k.$$

- Florea gave the expected asymptotic for the analogous fourth moment over the function field $\mathbb{F}_q[x]$ (where the Riemann hypothesis is known) with the base field $\mathbb{F}_q$ fixed and genus going to infinity.
- Our techniques should extend to give the asymptotic for $\mathcal{M}(4)$ unconditionally. This is work in progress by Shen and Stucky.

Moments of quadratic twists and rank

Let m_d be the order of vanishing of $L(s, f \otimes \chi_{8d})$ at $s = 1/2$, and let

$$R(X) = \sum_{\substack{0<8d<X \\ (d,2)=1}}^{*} m_d.$$

- Goldfeld proved that $R(X) \ll X$ conditionally on GRH.
- Trivially, $R(X) \ll X \log X$, while the work of Perelli and Pomykala gives $R(X) = o(X \log X)$.
- Our methods may yield $R(X) \ll X \log\log X$ proceeding along the same lines.
- Related: Mallesham Kummari is adapting these methods to derive asymptotics for moments of derivatives of the quadratic twist of modular L functions.

Reduction to Dirichlet polynomials

After an application of the approximate functional equation, we morally need to understand sums like

$$\underbrace{\sideset{}{^*}\sum_{m \asymp X} \left| \sum_{n \ll X} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right) \right|^2}_{L(1/2,\, f \otimes \chi_d)^2} . \tag{3}$$

Poisson and functional equation

We have two basic tools. Roughly,

- Poisson summation changes a sum like $\underbrace{\sum_{m \asymp M} \left(\frac{m}{n_1 n_2}\right)}$ into a dual sum of length $n_1 n_2 / M$. $\frac{\text{conductor}}{M}$
- Functional equation changes $\underbrace{\sum_{n \asymp N} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right)}$ into a similar dual sum of length $|m|^2/N$.

General rule of thumb is that we prefer shorter sums.
Neither tool seems to help us understand

$$\underbrace{\sum_{m \asymp X}^{*} \left| \sum_{n \asymp X} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right) \right|^2}.$$

$\frac{X^2}{X} = X$

$\frac{X^2}{X} = X$

Truncation

The functional equation and Poisson summation are useful in the easier range

$$\sum_{m \asymp X}^{*} \left| \overbrace{\sum_{n \ll X/(\log^A X)} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right)}^{\text{shorter}} \right|^2, \tag{4}$$

easier

for some large $A > 0$. Thus, the challenge is to bound sums of the form

$$S = \sum_{m \asymp X}^{*} \left| \sum_{\underbrace{n \asymp N}_{\text{close to } X.}} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right) \right|^2, \tag{5}$$

when N is close to X.

Large sieve type bound I

We will show that

$$\sum_{m \asymp X}^{*} \left| \sum_{n \asymp N} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right) \right|^2 \ll X, \tag{6}$$

which is best possible up to the implied constant. Assuming this, dyadic summation for $\frac{X}{\log^A X} \leq N \ll X$ gives the bound

$(\log\log X)^2$

$$\sum_{m \asymp X}^{*} \left| \sum_{X/(\log^A X) \ll n \ll X} \frac{\lambda_f(n)}{\sqrt{n}} \left(\frac{m}{n}\right) \right|^2 \ll X (\log X)^{\epsilon}.$$

This reduces the problem to considering (4).

Large sieve type bound II

The main challenge is to prove

$a(n) = \frac{\lambda_f(n)}{\sqrt{n}}$

$$S = \sideset{}{^*}\sum_{m \asymp X} \left| \sum_{n \asymp X} a(n) \left(\frac{m}{n}\right) \right|^2 \ll X, \tag{7}$$

which is best possible up to the implied constant.

$X(\log X)$

Heath-Brown's work implies that

more general $a(n)$

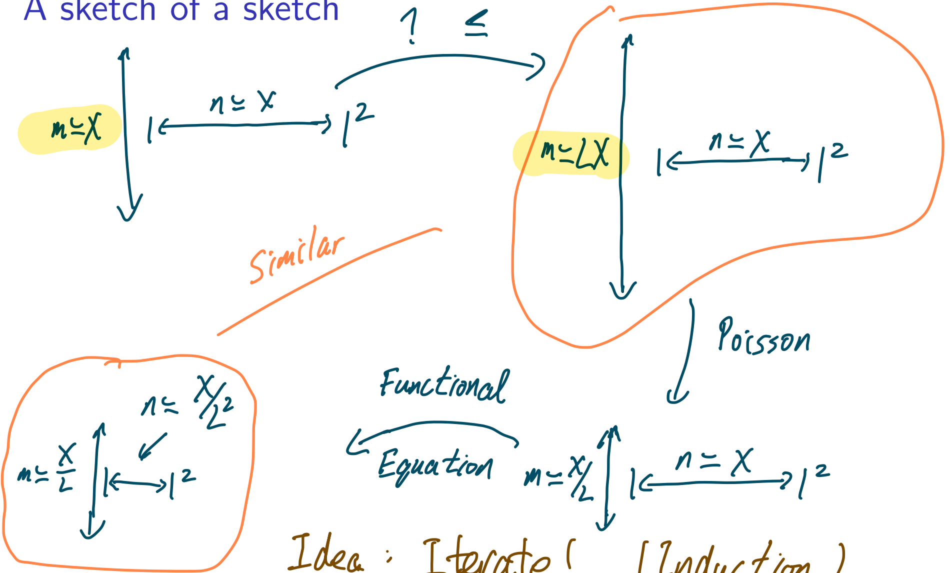
$$\sideset{}{^*}\sum_{m \asymp X} \left| \sum_{n \asymp X} a(n) \left(\frac{m}{n}\right) \right|^2 \ll X \cdot X^{\epsilon}, \tag{8}$$

and the work of Soundararajan and Young implicitly prove that

$$S \ll X(\log X)^{1/2+\epsilon}$$

conditionally on GRH.

A sketch of a sketch



Idea: Iterate! (Induction.)

Inflation I

$$\left(\frac{p^2}{n}\right) = \begin{cases} 1 & p \nmid n \\ 0 & p \mid n \end{cases}$$

For a prime $\underbrace{p \asymp \sqrt{L}}$

$$S = \sum_{m \asymp X}^{*} \left| \sum_{n \asymp X} a(n) \left(\frac{m}{n}\right) \right|^2 = \sum_{m \asymp X}^{*} \left| \underbrace{\sum_{\substack{n \asymp X \\ p \nmid n}} a(n) \left(\frac{mp^2}{n}\right)} + \underbrace{\sum_{\substack{n \asymp X \\ p \mid n}} a(n) \left(\frac{m}{n}\right)} \right|^2 .$$

Illustrative

Inflation II

Let $\mathcal{P}(L) \asymp \frac{\sqrt{L}}{\log L}$ be the number of primes in the interval $[\sqrt{L}, 2\sqrt{L}]$ and sum over all $p \in [\sqrt{L}, 2\sqrt{L}]$ to see that

$$\mathcal{P}(L)S \ll \sum_{\sqrt{L}\le p\le 2\sqrt{L}} \sum_{m\asymp X}^{*} \left| \sum_{n\asymp X} a(n) \left(\frac{mp^2}{n}\right) \right|^2 + \text{other}$$

$$\to \quad \le \sum_{m\asymp 4XL} \left| \sum_{n\asymp X} a(n) \left(\frac{m}{n}\right) \right|^2 + \text{other}.$$

$\{mp^2 : m \simeq X,\ m\ \square\text{-free},\ p \simeq \sqrt{L}\} \subseteq \{m : m \simeq XL\}$.

1. Positivity

2. $m_1 p_1^2 = m_2 p_2^2 \iff m_1 = m_2,\ p_1 = p_2$.

$m_i\ \square$-free

Remarks

- We have embedded our original sum over m into a longer sum, so that it is now advantageous to execute Poisson over m and begin the iterative process.
- Note that discarding the squarefree condition on m can be disastrous for arbitrary coefficients $a(n)$. We therefore expect to crucially use the special properties of $a(n) = \frac{\lambda_f(n)}{\sqrt{n}}$.
- We have used that $\left(\frac{p^2}{n}\right)$ tends to be trivial, and that the representation of m by $m'p^2$ is unique for m' squarefree.

Poisson

Opening up the square and applying Poisson summation roughly gives that

$$\sum_{m \asymp XL} \left| \sum_{n \asymp X} a(n) \left(\frac{m}{n} \right) \right|^2$$

$$= \underbrace{\mathcal{C}_f XL}_{\text{Diagonal term: } n_1 n_2 = \square} + \underbrace{\frac{XL}{2} \sum_{n_1, n_2 \asymp X} \frac{\lambda_f(n_1)\lambda_f(n_2)}{\sqrt{n_1 n_2} n_1 n_2} \sum_{\substack{k \neq 0 \\ k \ll X^2/XL \asymp X/L}} \overbrace{G_k(n_1 n_2)}^{\text{Gauss like sums}}}_{\text{Off diagonal term.}}.$$

Generically, $G_k(n_1 n_2)$ is $\chi_k(n_1 n_2)\sqrt{n_1 n_2}$ when $n_1 n_2$ is squarefree.

Functional equation

The sum over k is essentially restricted to $k \ll X^2/XL \asymp X/L$, so we need to bound

$$\frac{\cancel{X}L}{2\cancel{X}} \sum_{k \asymp X/L} \sum_{n_1, n_2 \asymp X} \frac{\lambda_f(n_1)\lambda_f(n_2)}{\sqrt{n_1 n_2}} \frac{G_k(n_1 n_2)}{\sqrt{n_1 n_2}}. \tag{9}$$

Now we replace $G_k(n_1 n_2)$ by $\chi_k(n_1 n_2)\sqrt{n_1 n_2}$ so we hope to instead study a quantity like

$$L \sum_{k \asymp X/L}^{*} \left| \sum_{n \asymp X} \frac{\lambda_f(n)\chi_k(n)}{\sqrt{n}} \right|^2. \tag{10}$$

Since the conductor $k \asymp X/L$ has been reduced, it now makes sense to apply the functional equation of $L(s, f \otimes \chi_k)$ to transform the sum over n to a sum of length X/L^2.

Structural comment

Diagonal Contribution

We expect

$$L \sum_{k \asymp X/L}^{*} \left| \sum_{n \asymp X/L^2} \frac{\lambda_f(n)\chi_k(n)}{\sqrt{n}} \right|^2 = CX \sum_{\substack{n_1, n_2 \asymp X/L^2 \\ n_1 n_2 = \square}} \frac{\lambda_f(n_1)\lambda_f(n_2)}{\sqrt{n_1 n_2}} + \text{small},$$

for some constant C.

- The "diagonal" contribution when $n_1 n_2$ is a perfect square dominates.
- However, generically $G_k(n_1 n_2) = 0$ when $n_1 n_2$ is not squarefree, so that the same "diagonal" contribution does not exist in the prior sum.
- Careful analysis of the factors at prime squares and higher powers is crucial.

"Many" prime squares

Our calculations suggest that

primes $\asymp \sqrt{L}$

$$\mathcal{P}(L)S \le CLX$$

for some constant C. Since $\mathcal{P}(L) \asymp \sqrt{L}/\log L$, $> L^{1/3+\varepsilon}$

$$S \le L^{2/3}X.$$

$\frac{1}{3} + \frac{1}{7} < \frac{1}{2}$

Using this as our induction hypothesis,

$$\mathcal{P}(L)S \le \underbrace{C_1XL}_{\text{Diagonal}} + \underbrace{C_2L(L^{2/3}X/L)}_{\text{Off-diagonal}} = (C_1 + C_2L^{-1/3})XL,$$

and so

$$S \le L^{2/3}X\boxed{\frac{C_1 + C_2L^{-1/3}}{L^{1/7}}} \le 1.$$

So the number of primes in the interval $[\sqrt{L}, 2\sqrt{L}]$ being large serves to control the loss of constant factors.

Open problems

$f \neq g$?

- $\sum_d^* L(1/2, f \otimes \chi_d) L(1/2, g \otimes \chi_d) \sim$ missing positivity
- $\sum_d^* L(1/2, f \otimes \chi_d)^2 (\text{Short Poly})^2 \sim$

  X^δ

  We have pretended that Poisson summation is useless unless the dual sum is shorter. But we have seen that the structure of the dual sum is different.