On the quality of the ABC-solutions

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Outline

- Introduction to the ABC-conjecture
- The quality of the ABC-solutions formed by the terms in Lucas sequences and associated Lucas sequences
- A new family of the ABC-solutions with quality > 1
- Laishram and Shorey's work on an explicit version of the ABC-conjecture
- Brocard-Ramanujan problem and proving Erdős conjecture under an explicit version of the ABC-conjecture

History

Joseph Oesterlé

David W. Masser

History

Definition

A triplet (a, b, c) is called an ABC-solution if $\gcd(a, b, c) = 1$ and

$$a + b = c.$$

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For an integer M let

$$\mathrm{rad}(M) = \prod_{p|M} p,$$

where p runs through the distinct prime factors of M. Set $\mathrm{rad}(1) = 1$.

- Example: $\mathrm{rad}(24) = 6$.

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The quality of an ABC-solutions

Definition

The *quality* of an ABC-solution, (a, b, c) is defined as

$$q(a,b,c) = \frac{\max\{\log|a|, \log|b|, \log|c|\}}{\log(\text{rad}(|abc|))}.$$

The polynomial case

Definition

Let $P_1,\ P_2$, and P_3 be three coprime polynomials in $\mathbb{C}[x]$ satisfying $P_1 + P_2 = P_3$. The quality of (P_1, P_2, P_3) is defined as

$$q(P_1, P_2, P_3) = \frac{\max\{\deg P_1, \deg P_2, \deg P_3\}}{\deg \operatorname{rad}(P_1 P_2 P_3)}.$$

Mason's Theorem

Theorem (Mason - 1984)

Let $P_1,\ P_2$, and P_3 be three coprime polynomials in $\mathbb{C}[x]$, not all of them constant, satisfying $P_1 + P_2 = P_3$. Then,

$$q(P_1, P_2, P_3) < 1.$$

Question

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Is 1 upper bound for the quality of ABC-solutions in $\mathbb{Z}$?

Lemma

For $k \geq 1$,

$$q(1, 3^{2^k} - 1, 3^{2^k}) > 1.$$

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Examples

ABC-solution	Quality
$2 + 3^{10} \cdot 109 = 23^5$	1.62991
$11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23$	1.62599
$19 \cdot 1307 + 7 \cdot 29^2 \cdot 31^8 = 2^8 \cdot 3^{22} \cdot 5^4$	1.62349
$283 + 5^{11} \cdot 13^2 = 2^8 \cdot 3^8 \cdot 17^3$	1.58076
$1 + 2 \cdot 3^7 = 5^4 \cdot 7$	1.56789

- ABC-solutions with quality ≥ 1.4 are called good ABC-solutions.

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- ABC-solutions with quality ≥ 1.4 are called good ABC-solutions.

The ABC-conjecture

Conjecture (Masser-Oesterlé, 1985)

Let $\epsilon > 0$. Then, there exist only finitely many ABC-solutions with the quality $\geq 1 + \epsilon$.

Another formulation for the ABC-conjecture

Conjecture (Masser-Oesterlé, 1985)

For any $\epsilon > 0$, there exists $K_\epsilon > 0$ such that,

$$\max\{|a|, |b|, |c|\} \leq K_\epsilon(\mathrm{rad}(|abc|))^{1+\epsilon}$$

for all the ABC-solutions (a, b, c).

Lucas sequences and associated Lucas sequences

Definition

Let P and Q be two coprime integers with the property that the equation $x^2 - Px + Q = 0$ has two roots α and β. Define

$$u_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad \text{and} \quad v_m = \alpha^m + \beta^m$$

for $m \geq 0$. The sequence $u = \{u_m\}_{m=0}^{\infty}$ and $v = \{v_m\}_{m=0}^{\infty}$ are called, respectively, the Lucas sequence and the associated Lucas sequence corresponding to the pair (P, Q).

Examples

Here is a table of Lucas sequences and associated Lucas sequences corresponding to the pair $(1, -1)$.

m	u_m(Fibonacci numbers)	v_m(Lucas numbers)
0	0	2
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11

Recall

Definition

The *quality* of an ABC-solution, (a, b, c) is defined as

$$q(a,b,c) = \frac{\max\{\log|a|, \log|b|, \log|c|\}}{\log(\mathrm{rad}(|abc|))}.$$

Relations in Lucas sequences and associated Lucas sequences

Lemma

Let $u = \{u_m\}_{m=0}^{\infty}$ *and* $v = \{v_m\}_{m=0}^{\infty}$ *be the Lucas sequence and the associated Lucas sequence corresponding to the pair* (P, Q). *Then,*

$$u_m^2 = u_{m-1}u_{m+1} + Q^{m-1}$$

and

$$v_m^2 = Du_m^2 + 4Q^m,$$

where $D = P^2 - 4Q$.

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The quality of the triplet $(u_m^2, u_{m-1}u_{m+1}, Q^{m-1})$

Consider the Lucas sequence corresponding to the pair $(1, -2)$.

m	$q(u_m^2, u_{m-1}u_{m+1}, Q^{m-1})$
29	0.658658188587
30	0.658936760063
31	0.659196603086
32	0.659439544838
33	0.659667182403
34	0.659880917741

The quality of the triplet $(u_m^2, u_{m-1}u_{m+1}, Q^{m-1})$

Theorem (B. 2022)

Assume the ABC-conjecture. Let $u = \{u_m\}_{m=0}^{\infty}$ be a Lucas sequence corresponding to the pair (P, Q) with $\gcd(P, Q) = 1$ and $D > 0$. Then,

$$\lim_{m\to\infty} q(u_m^2, u_{m-1}u_{m+1}, Q^{m-1}) = \frac{2}{3}.$$

Ideas of the proof

- Form the identity

$$\frac{1}{q} = \frac{\log \operatorname{rad}(|u_{m-1}u_m u_{m+1} Q|)}{\max\{\log|u_m^2|, \log|u_{m-1}u_{m+1}|, \log|Q^{m-1}|\}}.$$

- Let

$$A = \prod_{p^\alpha || \gamma(|u_{m-1}u_m u_{m+1} Q|)} p^{\alpha-1}$$

- Using properties of Lucas sequences we get

$$\lim_{m\to\infty} \frac{1}{q} = \frac{3}{2} - \lim_{m\to\infty} \frac{\log A}{\log|u_m^2|}$$

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Ideas of the proof

- Under the ABC-conjecture we have

$$\gamma(u_m) \ll_{\epsilon,|D|,|Q|} u_m^{\epsilon}.$$

- $\lim_{m\to\infty} \frac{\log A}{\log |u_m^2|} = 0.$

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The quality of the triplet $(v_m^2, Du_m^2, 4Q^m)$

Consider the Lucas sequence and associated Lucas sequence corresponding to the pair $(1, -2)$.

m	$q(v_m^2, Du_m^2, 4Q^m)$
27	1.0747313984
28	0.982456140539
29	0.983050847549
30	1.05091491125
31	0.984126984148
32	0.984615384626
33	1.00894232463

The quality of the triplet $(v_m^2, Du_m^2, 4Q^m)$

Theorem (B. 2022)

Assume the ABC-conjecture. Let $u = \{u_m\}_{m=0}^{\infty}$ and $v = \{v_m\}_{m=0}^{\infty}$ be a Lucas sequence and an associated Lucas sequence corresponding to the pair (P, Q), where $\gcd(P, Q) = 1$ and $D > 0$. Then,

$$\lim_{m\to\infty} q(v_m^2, Du_m^2, 4Q^m) = 1$$

when $(v_m^2, Du_m^2, 4Q^m)$ is an ABC-solution and

$$\lim_{m\to\infty} q(v_m^2/4, Du_m^2/4, Q^m) = 1$$

when $(v_m^2/4, Du_m^2/4, Q^m)$ is an ABC-solution.

A new family of ABC-solutions with quality > 1

Theorem (B. 2022)

Let $u = \{u_m\}_{m=0}^{\infty}$ and $v = \{v_m\}_{m=0}^{\infty}$ be the Lucas sequence and the associated Lucas sequence corresponding to the pair (P, Q), with an odd discriminant. If $Q = -2^{\alpha}$, where $\alpha \geq 0$ or Q is a negative square-full odd integer, then either

$$q(v_m^2, Du_m^2, 4Q^m) > 1$$

or

$$q(v_m^2/4, Du_m^2/4, Q^m) > 1,$$

where $m = (2k+1)\mathrm{rad}D$ for $k > 0$.

Key Lemma

Lemma

Let $u = \{u_m\}_{m=0}^{\infty}$ be the Lucas sequence corresponding to the pair (P, Q). Then,

$$\mathrm{rad}(D) | u_{\mathrm{rad}(D)},$$

where D is the discriminant of u.

Some examples

(P, Q)	$\text{rad}D$	ABC-solution	quality
$(1, -2)$	3	$32 + 49 = 81$	1.17
$(3, -4)$	5	$1046529 + 4096 = 1050625$	1.071
$(1, -1)$	5	$121 + 4 = 125$	1.027
$(9, -23^2)$	13	$a + b = c$	1.013

where $a = (3^2 \cdot 79 \cdot 131 \cdot 313 \cdot 1301 \cdot 165352513)^2$, $b = 2^2 \cdot 23^{26}$, and $c = 13^3 \cdot (13 \cdot 53 \cdot 591553 \cdot 332494553)^2$.

Transfer method

Lemma

Let (a, b, c) *be an* ABC*-solution with the quality* > 1*. Then,*

$$(b-a)^2 + 4ab = c^2$$

and the quality of this ABC*-solution is greater than* 1*.*

Transfer methods

From

$$32 + 49 = 81$$

we can build

$$17^2 + 2^7 \cdot 7^2 = 3^8$$

which is an ABC-solution with quality 1.337.

Notation

- For any ABC-solution (a, b, c) set $N = \mathrm{rad}(abc)$.
- We also show the number of distinc prime factors of N with ω.
  Example: $N = 24 = 2^3 \cdot 3$, $\omega = 2$.

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- For any ABC-solution (a, b, c) set $N = \mathrm{rad}(abc)$.
- We also show the number of distinc prime factors of N with ω.
  Example: $N = 24 = 2^3 \cdot 3$, $\omega = 2$.

Explicit ABC-conjecture

Conjecture (A. Baker, 2004)

Let a, b, and c be positive coprime integers satisfying

$$a + b = c.$$

Then,

$$c < \frac{6}{5} N \frac{(\log N)^{\omega}}{\omega!},$$

where $N = \mathrm{rad}(abc)$ and $\omega = \omega(N)$.

Explicit ABC-conjecture

We can show that for given $\epsilon > 0$ there exists a constant $K_\epsilon > 0$, such that

$$\frac{(\log N)^\omega}{\omega!} < K_\epsilon N^\epsilon,$$

where $\omega = \omega(N)$.

Laishram and Shorey's work (2012)

- Assuming Baker's explicit version they showed that for $\epsilon = 0.75$ we have

$$\frac{6(\log N)^{\omega}}{5\ \omega!} < N^{0.75},$$

  for all integers N.
- Therefore, $q(a,b,c) < 1 + 0.75$ for any ABC-solution where a, b, and c are three positive coprime integers and $N = \mathrm{rad}(abc)$.
- In 2013, Silva reduced ϵ to $\epsilon = 0.738$ by generalizing their method.

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Refinement on Laishram and Shorey's work

Theorem (B. 2022)

Assume the explicit ABC-conjecture. Let a, b, and c be three positive coprime integers satisfying $a + b = c$ and set $\mathrm{rad}(abc) = N$. Then,

$$c < N^{1.71}.$$

Brocard-Ramanujan problem

Question

Are there more solutions to

$$n! + 1 = m^2$$

except $(n, m) = (4, 5)$, $(5, 11)$, $(7, 71)$?

Erdős' conjecture

- Erdős conjectured that these are the only solutions.
- A joint work of Berndt and Galway shows that up to $n < 10^9$ these are the only solutions.
- In 1993, Overholt proved the finiteness of the solutions assuming the ABC-conjecture but his results does not provide an explicit upper bound on n or m.

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Erdős' conjecture

Theorem (B. 2022)

Under the explicit ABC-conjecture, the pairs $(4,5)$, $(5,11)$, and $(7,71)$ are the only solutions to $n! + 1 = m^2$.

Erdős' conjecture

Lemma (Chebyshev's upper bound)

For any integer n,

$$\mathrm{rad}(n!) = \prod_{p \le n} p$$

and

$$\prod_{p \le n} p < 4^n.$$

Lemma (Stirling's lower bound)

For any integer n,

$$\sqrt{2\pi n}(n/e)^n e^{1/12n+1} \le n!.$$

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Thank you for listening !