Geometric Langlands for Hypergeometric Sheaves

Joint with Lingfei Yi

Reference :

Kamgarpour — Yi :
"Geometric Langlands for hypergeometric sheaves"
arXiv : 2006.10870

Heinloth – Ngo – Yun :
"Kloosterman sheaves for reductive groups"
Annals of Math. 2013

Yun : "Rigidity for automorphic representations
and local systems"
Current Development in Math. 2014

Note : Lingfei Yi is a student of Xinwen Zhu at Caltech and will be applying for a postdoc in November.

I. Overview

- X smooth proj. geom. connected curve / field k.
- G reductive group over $k(X)$.
- $\check{G}$ Langlands dual group

(In a few min $X = \mathbb{P}^1$, $G = GL_n \Rightarrow \check{G} = GL_n$)
$k = \mathbb{F}_q$.

- Goal of geometric Langlands is to establish a duality

$$\mathrm{Bun}_G(X) \longleftrightarrow \mathrm{LocSys}_{\check{G}}(X).$$

- Core Conjecture: Let $S \subseteq X$ be a finite set.

For every irreducible $\check{G}$-local system E on $X - S$, there exists a (non-zero) perverse sheaf $\mathcal{A} = \mathcal{A}_E$ on the moduli of G bundles on X, equipped with level structure at S, whose Hecke eigenvalue is E.

Status report: We know a lot if E is unramified; i.e. extends to a loc. system on X.

We know very little if E is ramified..

↳ See introduction of our arXiv preprint.

Theorem: Core Conjecture holds for all (generalised) irreducible hypergeometric local systems.

★ We prove this by explicitly constructing the desired eigensheaves.

"Local system" can have many meanings; e.g.

$k = \mathbb{F}_q \rightsquigarrow$ lisse ℓ-adic sheaf $(\ell \neq p)$

$\rightsquigarrow$ Overconvergent F-isocrystal

$k = \mathbb{C} \rightsquigarrow$ vector bundle equipped with (flat) connection.

Our theorem applies to all these settings.

To fix ideas, we work with ℓ-adic sheaves over finite fields.

II. Hypergeometric Local Systems

The notion of local system originated in Riemann's seminal work on the Euler–Gauss hypergeometric function. Riemann's revolutionary insight was that one can (and should) study the "local system" of holomorphic solutions of the hypergeometric diff. equation on $\mathbb{P}^1 - \{0, 1, \infty\}$.

Using this approach, Riemann recovered Gauss & Kummer relations for hypergeom. function with almost no computation.

Riemman's investigation was a stunning success because the hypergeom. local system is rigid; i.e. uniquely determined as soon one speifies its monodromy at $0, 1, \infty$.

The ℓ-adic analogue of (generalized) hypergeom. loc. systems were defined by Katz in 1990. We will call them <u>hypergeometric sheaves</u>.

To specify them, we need the following <u>hypergeometric initial data</u>:

(i) A finite field k, $\mathrm{char}(k) \neq \ell$.

(ii) An additive char. $\psi : k \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

(iii) Integers (m,n) with $0 \leq m < n$ and mult. characters

$\chi_1, \ldots, \chi_n$
$\rho_1, \ldots, \rho_m : k^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

$$\mathbb{G}_a \times \mathbb{G}_m^n \times \mathbb{G}_m^m \xleftarrow{\ p\ } \mathbb{G}_m^n \times \mathbb{G}_m^m \xrightarrow{\ q\ } \mathbb{G}_m$$

$$\Big(\sum x_i - \sum y_j,\ x_1, \ldots, x_n, y_1^{-1}, \ldots, y_m^{-1}\Big) \longmapsfrom (x_1, \ldots, x_n, y_1, \ldots, y_m) \longmapsto (x_1 \cdots x_n)(y_1 \cdots y_m)^{-1}$$

$$\mathcal{H} = \mathcal{H}(\psi, \chi_1, \ldots, \chi_n, \rho_1, \ldots, \rho_m) :=$$

$$q_! p^* (\mathcal{L}_\psi \boxtimes \mathcal{L}_{\chi_1} \boxtimes \cdots \boxtimes \mathcal{L}_{\chi_n} \boxtimes \mathcal{L}_{\rho_1} \boxtimes \cdots \boxtimes \mathcal{L}_{\rho_m})[n+m-1]$$

Here, $\mathcal{L}_\psi$ is the rank one local system on $\mathbb{G}_a$ with Frob. trace ψ (Artin – Schrier sheaf)

$\mathcal{L}_{\chi_i}$, $\mathcal{L}_{\rho_j}$ are one dim. local systems on $\mathbb{G}_m$ with Frob. trace χ_i and ρ_j (Kummer sheaf)

Theorem (Katz): ① $\mathcal{H}[1]$ is perverse.

② $\mathcal{H}[1]$ is irred. iff $\chi_i \neq \rho_j$ $\forall i,j$.

(We say χ_i, ρ_j are disjoint)

③ $m = n \Longrightarrow \mathcal{H}$ is a local system on $\mathbb{P}^1 - \{0,1,\infty\}$ with tame ramification at $0, 1, \infty$ and pseudo-reflection monodromy at 1. $\rightsquigarrow \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \lambda \end{pmatrix}$

④ $m < n \Longrightarrow \mathcal{H}$ is lisse on $\mathbb{P}^1 - \{0, \infty\}$ with tame singularity at 0 and wild one at ∞.

⑤ If $\mathcal{H}$ is irred. then it is rigid.

[Katz] Exponential Sums and Diff. Equations Annals of Maths Studies 1990.

PLAN

The rest of the talk will be about constructing the eigensheaf corresponding to a tame hypergeometric sheaf $\mathcal{H} = \mathcal{H}(\psi, \chi_1, \dots, \chi_n, \rho_1, \dots, \rho_m)$.

I will not talk about what is a ramified Hecke eigensheaf (see §5 of our article).

- $\boxed{\text{Henceforth } G = GL_n \ , \ X = \mathbb{P}^1}$ $k = \mathbb{F}_q$
- $\mathcal{O}_x$ = completed local ring at $x \approx k[[s]]$
- $\mathfrak{p}_x$ = maximal ideal of $\mathcal{O}_x \approx sk[[s]]$

III. Group scheme controlling $\mathcal{H}$

- Define a group scheme $\mathcal{G}/X$ which is isom. to G on $\mathbb{P}^1 - \{0,\infty\}$, and satisfies

- $$\mathcal{G}(\mathcal{O}_x) = \begin{cases} I(1) & \text{if } x = 0 \\ Q(1) & \text{if } x = 1 \\ I(1) & \text{if } x = \infty \end{cases}$$

Here I is the Iwahori and $I(1)$ its prounipotent radical.

- Example: $G = GL_3$, $I = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ p & \mathcal{O}^\times & \mathcal{O} \\ p & p & \mathcal{O}^\times \end{pmatrix} \xrightarrow{S \mapsto 0} B$

- $$I(1) = \begin{pmatrix} 1+p & \mathcal{O} & \mathcal{O} \\ p & 1+p & \mathcal{O} \\ p & p & 1+p \end{pmatrix} \xrightarrow{S \mapsto 0} U$$

This explains the structure at 0 and ∞.

The group $Q(1)$ is defined using the mirabolic.

Recall the mirabolic is defined by

$$\overline{Q} = \begin{pmatrix} GL_{n-1} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ 0 \cdots 0 & GL_1 \end{pmatrix} \subseteq GL_n(K)$$

$$\overline{Q(1)} = \begin{pmatrix} \boxed{0} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \boxed{0 \cdots 0} & 1 \end{pmatrix}$$

Now define $\mathrm{Lie}(Q) = \left(\begin{array}{ccc|c} \mathfrak{g} & \cdots & \mathfrak{g} & \mathfrak{g} \\ & & \vdots & \vdots \\ \mathfrak{g} & \cdots & \mathfrak{g} & \mathfrak{g} \\ \hline p & \cdots & p & \mathfrak{g} \end{array}\right)$ $\quad s_1 \mapsto 0 \rightsquigarrow \mathrm{Lie}(\overline{Q})$

$$\mathrm{Lie}(Q(1)) = \left(\begin{array}{ccc|c} p & \cdots & p & \mathfrak{g} \\ \vdots & & \vdots & \vdots \\ p & \cdots & p & \mathfrak{g} \\ \hline p & \cdots & p & p \end{array}\right) \quad s_1 \mapsto 0 \rightsquigarrow \mathrm{Lie}(\overline{Q}(1))$$

$*\ \mathcal{G}$ has a cousin $\mathcal{G}'$ defined by $\mathcal{G}'|_{\mathbb{P}^1 - \{0,1,\infty\}} \simeq G$

and $\mathcal{G}' = \begin{cases} I & x = 0 \\ Q & x = 1 \\ I & x = \infty \end{cases}$

IV. Bun_G

Let Bun_G = moduli of G-bundles on X

= moduli of right G-torsors on X.

Similarly, we have $\mathrm{Bun}_{G'}$.

Concretely: $\mathrm{Bun}_{G'}$ = moduli of rank n-V.B. on X

+ full flag at 0

+ flag of type Q at 1

+ full flag at ∞

Bun_G = moduli of bundles + flags as above

+ additional vectors in the graded

pieces of the flag.

See § 4.1 of arXiv preprint.

We have forgetfull maps

$$
\begin{tikzcd}
\mathrm{Bun}_{\mathcal{G}} \arrow[r] & \mathrm{Bun}_{\mathcal{G}'} \arrow[r] & \mathrm{Bun}_n
\end{tikzcd}
$$

principal bundle $\underbrace{I/I(1) \times Q/Q(1) \times I/I(1)}_{H = \mathcal{G}'/\mathcal{G}}$

$G/I \times G/Q \times G/I$ fibration.

inducing isom.

$$\pi_0(\mathrm{Bun}_{\mathcal{G}}) \simeq \pi_0(\mathrm{Bun}_{\mathcal{G}'}) \simeq \pi_0(\mathrm{Bun}_n) \underset{\deg}{\simeq} \mathbb{Z}$$

For each $\alpha \in \mathbb{Z}$, let $\mathrm{Bun}_{\mathcal{G}}^{\alpha}$ be the corresponding connected component of $\mathrm{Bun}_{\mathcal{G}}$.

V. The Pair $(H, \mathcal{C})$

We saw in the previous section that the group

$H = I/_{I(1)} \times Q/_{Q(1)} \times I/_{I_\infty}$ acts on Bun_G

Now $(\chi_1, \dots, \chi_n)$ defines a character

$$I \longrightarrow I/_{I(1)} \simeq (K^\times)^n \xrightarrow[\chi_1 \cdots \chi_n]{} \overline{\mathbb{Q}}_\ell^\times$$

Similarly, we have the character $\rho = \rho_1 \cdots \rho_n : I \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

Thus, we have a character

$$c : H \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$
$$(a, b, c) \longmapsto \chi(a)\, \rho(c)$$

Let $\mathcal{C}$ be the rank one local system on H whose trace function is c.

This is an example of a character sheaf.

VI. Outline of the Proof

Our main theorem is proved in three steps:

① There exists a unique $(\mathcal{H}, e)$-equiv. irred. perverse sheaf $\mathcal{A}_\alpha$ on Bun_g^α.

② The perverse sheaf $\mathcal{A} = (\mathcal{A}_\alpha)_{\alpha \in \mathbb{Z}}$ is a Hecke eigensheaf on Bun_g.

③ The Hecke eigenvalue is $\mathcal{H}$.

① is the key statement here.

② follows from ① + main theorem of [Yun], [HNY]
See §6 of arXiv preprint

③ Can be proved by computing trace functions.
(But tedious)
See §4 of preprint.

How to prove ①?

An H-orbit on $\mathrm{Bun}_{\mathfrak{g}}$ is called relevant if it supports a $(H, \mathcal{L})$-equivariant sheaf.

$\Longleftrightarrow$ $\mathcal{L}\big|_{\mathrm{Stab}_H(\mathbb{O})^\circ}$ is trivial.

Key fact: There exists a unique relevant orbit on each component $\mathrm{Bun}_{\mathfrak{g}}^{\alpha}$.

The proof involves intricate combinatorics of $\mathrm{Bun}_{\mathfrak{g}}$. (See §8 of preprint).