Height gaps for coefficients of D-finite power series

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Height

There is a general theory of Weil heights for points on projective varieties over $\bar{\mathbb{Q}}$.

For most of this talk, we only need the absolute logarithmic Weil height

$$h: \bar{\mathbb{Q}} \to \mathbb{R}_{\geq 0}.$$

In general, one needs to combine the contributions from all absolute values to define height functions. But for the above h on $\bar{\mathbb{Q}}$, there is an alternative explicit formula, as follows.

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Fix an embedding $\bar{\mathbb{Q}} \subset \mathbb{C}$. For $\alpha \in \bar{\mathbb{Q}}$, express its minimal polynomial over $\mathbb{Z}$ as:

$$c(x - \alpha_1) \cdots (x - \alpha_d).$$

Then $h(\alpha) = \dfrac{1}{d}\left(\log|c| + \sum_{i=1}^{d} \log \max\{|\alpha_i|, 1\}\right)$.

Example: $\alpha \in \mathbb{Q}$, express $\alpha = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$, then $h(\alpha) = \log \max\{|a|, |b|\}$.

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More example/property: let $P(x) \in \bar{\mathbb{Q}}[x]$ of degree D, then

$$h(P(\alpha)) = Dh(\alpha) + O(1)\ \forall \alpha \in \bar{\mathbb{Q}}.$$

This means there exists $C > 0$ depending only on $P(x)$ such that

$$|h(P(\alpha)) - Dh(\alpha)| \leq C\ \forall \alpha \in \bar{\mathbb{Q}}.$$

In particular $h(P(n)) = D \log |n| + O(1)$ for $n \in \mathbb{Z}$.

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D-finite series

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, K is a field, and $m \in \mathbb{N}$.

Let $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}_0^m$ and let $\mathbf{x} = (x_1, \ldots, x_m)$ be the vector of the indeterminates $x_1, \ldots, x_m$. Write $\mathbf{x}^{\mathbf{n}}$ to denote the monomial $x_1^{n_1} \ldots x_m^{n_m}$ having the total degree

$$\|\mathbf{n}\| := n_1 + \ldots + n_m.$$

Write $\dfrac{\partial^{\|\mathbf{n}\|}}{\partial \mathbf{x}^{\mathbf{n}}}$ to denote the operator

$$\left(\frac{\partial}{\partial x_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{n_m}$$

on $K[[\mathbf{x}]] := K[[x_1, \ldots, x_m]]$.

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D-finite series

A power series $f(\mathbf{x}) \in K[[\mathbf{x}]]$ is said to be D-finite (over $K(\mathbf{x})$) if all the derivatives $\dfrac{\partial^{\|\mathbf{n}\|} f}{\partial \mathbf{x}^{\mathbf{n}}}$ for $\mathbf{n} \in \mathbb{N}_0^m$ span a finite-dimensional vector space over $K(\mathbf{x})$.

Problem: $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$ is D-finite, study the growth of $h(a_{\mathbf{n}})$ with respect to $\|\mathbf{n}\|$.

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Examples

It is helpful to think of the univariate case ($m = 1$), here $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \bar{\mathbb{Q}}[[x]]$ is D-finite iff it satisfies a linear differential equation with coefficients in $\bar{\mathbb{Q}}[x]$.

Equivalently, the coefficients (eventually) satisfy a linear recurrence relation with polynomial coefficients: there exist $d \in \mathbb{N}$ and $P_0(x), \ldots, P_d(x) \in \bar{\mathbb{Q}}[x]$ with $P_d \neq 0$ such that

$$P_d(n)a_{n+d} + \ldots + P_0(n)a_n = 0$$

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Example 1: exponential function

$$f(x) = \sum \frac{x^n}{n!}, \ h(a_n) = \log(n!) \sim n \log n.$$

Example 2: rational function with at least one pole not being a root of unity

$$f(x) = \frac{1}{1-2x} = \sum 2^n x^n, \ h(a_n) = n \log 2.$$

Example 3: logarithmic function

$$f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots, \ h(a_n) = \log n.$$

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Example 4: rational function with at least one pole of order at least 2

$$f(x) = \frac{1}{(1-x)^2} = \sum nx^{n-1}, \ h(a_n) = \log(n+1).$$

Example 5: rational function in which the a_n's belong to a finite set

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A height gap result in 2019

Below $f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}_0^m} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}} \in \bar{\mathbb{Q}}[[\mathbf{x}]]$ is a D-finite power series in m variables.

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Theorem (Bell-N.-Zannier)

Suppose $\lim_{\|\mathbf{n}\|\to\infty} \dfrac{h(a_{\mathbf{n}})}{\log \|\mathbf{n}\|} = 0$. *Then:*

(a) *f is a rational function.*

(b) *If f is not a polynomial, its denominator, up to scalar multiplication, has the form*

$$\prod_{i=1}^{\ell}(1-\zeta_i \mathbf{x}^{\mathbf{n}_i})$$

where $\ell \geq 1$, ζ_i is a root of unity, $\mathbf{n}_i \in \mathbb{N}_0^m \setminus \{0\}$ for $1 \leq i \leq \ell$, and the $1-\zeta_i \mathbf{x}^{\mathbf{n}_i}$'s are ℓ distinct irreducible polynomials.

(c) *The coefficients $(a_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}_0^m}$ belong to a finite set.*

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A height gap result in 2019

Key observation after the above result: there's a "gap" in the possible growth of $h(a_{\mathbf{n}})$. More precisely if $h(a_{\mathbf{n}})$ is dominated by $\log\|\mathbf{n}\|$ then it is $O(1)$.

Many results motivated by the above theorem

For the rest of this talk: focus on **univariate power series** $f(x) = \sum a_n x^n \in \bar{\mathbb{Q}}[[x]]$.

From the previous 5 examples, it's natural to ask whether we can completely classify the growth of $h(a_n)$ as $O(n \log n)$, $O(n)$, $O(\log n)$, or $O(1)$ when f is D-finite. A more precise open problem will be stated at the end of the talk.

Right after our work in 2019, we have some idea for further results toward the above classification. But its release was delayed until June 2022! In the meantime, there are highly interesting results motivated by our work.

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Many results motivated by the above theorem

- Results by Bell, Hu, Ghioca, Satriano on a height gap phenomenon in arithmetic dynamics.
- A complete classification for the possible height growth of coefficients of Mahler functions by Adamczewski and Bell.
- Dimitrov's beautiful proof of the Schinzel-Zassenhauss conjecture from the 1960s.

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Our result in June 2022

A set $S \subseteq \mathbb{N}$ is said to have positive upper density if

$$\limsup \frac{|S \cap [1, n]|}{n} > 0,$$

otherwise S is said to have zero density.

Let $\alpha \in \bar{\mathbb{Q}}$, its denominator $\mathrm{den}(\alpha)$ is the smallest $d \in \mathbb{N}$ such that $d\alpha$ is an algebraic integer.

In the next theorem: $K \subset \mathbb{C}$ is a number field, $f(x) = \sum a_n x^n \in K[[x]]$ is D-finite, $r \in [0, \infty]$ is the radius of convergence of f.

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Theorem (Bell-N.-Zannier)

(a) *If $r \in \{0, \infty\}$ and f is not a polynomial then $h(a_n) = O(n \log n)$ for every large n and $h(a_n) \gg n \log n$ on a set of positive upper density.*

(b) *If $r \notin \{0, \infty\}$ then at least one of the following holds:*

   (i) *$h(a_n) \gg n$ on a set of positive upper density.*

   (ii) *$\mathrm{den}(a_n) \gg n$, and hence $h(a_n) > (\log n)/[K : \mathbb{Q}] + O(1)$ on a set of positive upper density.*

   (iii) *f is a rational function whose poles are roots of unity.*

Some open problems

Roughly speaking, the previous theorem says that $n \log n$, n, $\log n$, and the constant function are the possible lower bounds for $h(a_n)$.

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Question

$f(x) = \sum a_n x^n \in \bar{\mathbb{Q}}[[x]]$ is D-finite. Is it true that one of the following holds?

(i) $h(a_n) = O(n \log n)$ *for every* n *and* $h(a_n) \gg n \log n$ *on a set of positive upper density.*

(ii) $h(a_n) = O(n)$ *for every* n *and* $h(a_n) \gg n$ *on a set of positive upper density.*

(iii) $h(a_n) = O(\log n)$ *for every* n *and* $h(a_n) \gg \log n$ *on a set of positive upper density.*

(iv) $h(a_n) = O(1)$ *for every* n*.*

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Here's a weaker version of the above.

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$f(x) = \sum a_n x^n \in \bar{\mathbb{Q}}[[x]]$ is D-finite. Is it true that the following hold?

(i) $h(a_n) = O(n \log n)$ *for every n.*

(ii) *If* $h(a_n) = o(n \log n)$ *then* $h(a_n) = O(n)$.

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Remark: parts (i) and (iv) are already known from our result in 2019.

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Some open problems

Part (ii) above is analogous to a long standing open problem in the theory of Siegel E-functions. Instead of $h(a_n)$, the below problem considers the (affine) height of a tuple of algebraic numbers.

Question

$f(x) = \sum a_n x^n \in \bar{\mathbb{Q}}[[x]]$ is D-finite. Assume that $h(a_0, \ldots, a_n) = o(n \log n)$. Is it true that $h(a_0, \ldots, a_n) = O(n)$?

Some idea for the proof

The essential case to consider is:

- $f(x) \in \mathbb{Q}[[x]]$ is D-finite with rational coefficients.
- Its radius of convergence $r = 1$.

And we need to prove that at least one of the following holds:

A. $\mathrm{den}(a_n) \gg n$ on a set of positive upper density.

B. f is rational.

Suppose A is not true. This means that for a large N, there is a "thin" exceptional subset E of $\{1, \ldots, N\}$ such that $\mathrm{den}(a_n)$ is small vs n for every $n \in \{1, \ldots, N\} \setminus E$. We need to prove that f is rational.

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Some idea for the proof

First ingredient: Hankel determinant and rational approximation.

Let $g(x) = \sum b_n x^n$ and $m \geq 0$, define

$$\Delta_m(g) = \det \begin{pmatrix} b_0 & b_1 & \dots & b_m \\ b_1 & b_2 & \dots & b_{m+1} \\ \dots & & & \\ b_m & b_{m+1} & \dots & b_{2m} \end{pmatrix}.$$

First ingredient: Hankel determinant and rational approximation.

Let $g(x) = \sum b_n x^n$ and $m \geq 0$, define

$$\Delta_m(g) = \det \begin{pmatrix} b_0 & b_1 & \dots & b_m \\ b_1 & b_2 & \dots & b_{m+1} \\ \dots & & & \\ b_m & b_{m+1} & \dots & b_{2m} \end{pmatrix}.$$

Some idea for the proof

Facts:

- If $\Delta_m(g) = 0$ for many consecutive values of m then g can be "well" approximated by rational functions.
- If a D-finite power series can be well approximated by a rational function then it is indeed a rational function.

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Second ingredient: Polya's inequality.

Suppose $g(x) = \sum b_n x^n \in \mathbb{C}[[x]]$ converges in the open unit disk and can be continued analytically beyond the open unit disk. Then there exists $\rho < 1$ such that

$$|\Delta_m(g)| < \rho^{m^2}$$

for all large m.

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Third ingredient: construction of an auxiliary polynomial.

Recall that we assume Property A does not hold. This means for a large N, there's a thin subset E of $\{1, \ldots, N\}$ such that $\mathrm{den}(a_n)$ is small vs n for $n \in \{1, \ldots, N\} \setminus E$.

Construct an integer-valued polynomial P such that $P(n) = 0$ for $n \in E$. Hence although $\mathrm{den}(a_n)$ for $n \in E$ might be large, we simply have $P(n)a_n = 0$.

Then consider:

$$g(x) := \sum P(n)a_n x^n$$

which is a linear combination of the derivatives of $f(x)$.

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By Polya's inequality $|\Delta_m(g)| < \rho^{m^2}$ for some $\rho < 1$.

On the other hand, $\Delta_m(g)$ is a rational number whose denominator is quite small.

Therefore $\Delta_m(g) = 0$. Then we can have that g is rational and it's not hard to prove rationality of f from here.

This is just a rough idea. We need to make precise all the involving estimates and construct P carefully so that everything works.

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THANK YOU!