An analogue of Mertens function for the Rankin–Selberg *L*-function

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Definition (Generalized upper half-plane $\mathcal{H}^n$)

Let $n \geq 2$. The generalized upper half-plane $\mathcal{H}^n$ associated to $GL(n,\mathbb{R})$ is defined to be the set of all $n \times n$ matrices of the form $z = x \cdot y$ where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix},$$

$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i > 0$ for $1 \leq i \leq n-1$.

Definition (Maass form)

Let $n \geq 2$, and let $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$. A Maass form for $SL(n,\mathbb{Z})$ of type v is a smooth function $f \in \mathcal{L}^2(SL(n,\mathbb{Z})\backslash\mathcal{H}^n)$ which satisfies

1. $f(\gamma z) = f(z)$, for all $\gamma \in SL(n,\mathbb{Z}), z \in \mathcal{H}^n$,
2. $Df(z) = \lambda_D f(z)$, for all $D \in \mathfrak{D}^n$ where $\mathfrak{D}^n$ is the center of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{gl}(n,\mathbb{R})$ is the Lie algebra of $GL(n,\mathbb{R})$,
3. $$\int_{(SL(n,\mathbb{Z})\cap U)\backslash U} f(uz)du = 0,$$
   for all upper triangular groups U of the form
   $$U = \left\{ \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & & * \\ & & \ddots & \\ & & & I_{r_b} \end{pmatrix} \right\},$$
   with $r_1 + r_2 + \cdots + r_b = n$. Here I_r denotes the $r \times r$ identity matrix, and $*$ denotes arbitrary real entries.

A *Hecke–Maass form* is a Maass form which is an eigenvector for all the Hecke operators.

Let $f(z)$ be a Hecke–Maass form of type $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$ for $SL(n, \mathbb{Z})$. Then it has the Fourier expansion

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{j=1}^{n-1} \left|m_j\right|^{\frac{j(n-j)}{2}}} \times W_J \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v, \psi_{1,\ldots,1,\frac{m_{n-1}}{\left|m_{n-1}\right|}} \right),$$

where

$$M = \begin{pmatrix} m_1 \cdots m_{n-2} \cdot \left|m_{n-1}\right| & & & & \\ & \ddots & & & \\ & & m_1 m_2 & & \\ & & & m_1 & \\ & & & & 1 \end{pmatrix},$$

$$A(m_1, \ldots, m_{n-1}) \in \mathbb{C}, \qquad A(1, \ldots, 1) = 1,$$

$$\psi_{1,\ldots,1,\epsilon}\left(\begin{pmatrix} 1 & u_{n-1} & & & \\ & 1 & u_{n-2} & & * \\ & & \ddots & \ddots & \\ & & & 1 & u_1 \\ & & & & 1 \end{pmatrix}\right) = e^{2\pi i(u_1 + \cdots + u_{n-2} + \epsilon u_{n-1})},$$

$U_{n-1}(\mathbb{Z})$ denotes the group of $(n-1) \times (n-1)$ upper triangular matrices with 1s on the diagonal and an integer entry above the diagonal and W_J is the Jacquet Whittaker function.

Definition (Dual Maass form)

If $f(z)$ is a Maass form of type $(v_1, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$, then

$$\tilde{f}(z) := f(w \cdot (z^{-1})^T \cdot w),$$

$$w = \begin{pmatrix} & & & (-1)^{\left[\frac{n}{2}\right]} \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

is a Maass form of type $(v_{n-1}, \ldots, v_1)$ for $SL(n, \mathbb{Z})$ called the *dual Maass form*.

If $A(m_1, \ldots, m_{n-1})$ is the $(m_1, \ldots, m_{n-1})$-Fourier coefficient of f, then $A(m_{n-1}, \ldots, m_1)$ is the corresponding Fourier coefficient of $\tilde{f}$.

Definition (Godement–Jacquet L-function)

The Godement–Jacquet L-function $L_f(s)$ attached to f is defined for $\Re(s) > 1$ by

$$L_f(s) = \sum_{m=1}^{\infty} \frac{A(m,1,\ldots,1)}{m^s} = \prod_p \prod_{i=1}^{n} (1 - \alpha_{p,i} p^{-s})^{-1},$$

where the $\{\alpha_{p,i}\}, 1 \leq i \leq n$ are the complex roots of the monic polynomial

$$X^n + \sum_{r=1}^{n-1} (-1)^r A(\overbrace{1,\ldots,1}^{r-1 \text{ terms}}, p, 1, \ldots, 1) X^{n-r} + (-1)^n \in \mathbb{C}[X], \quad \text{and}$$

$$A(\overbrace{1,\ldots,1}^{r-1}, p, 1, \ldots, 1) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \alpha_{p,i_1} \ldots \alpha_{p,i_r}, \qquad \text{for } 1 \leq r \leq n-1.$$

Rankin–Selberg L-function

For $n \geq 2$, let f, g be two Maass forms for $SL(n, \mathbb{Z})$ of type $v_f, v_g \in \mathbb{C}^{n-1}$, respectively, with Fourier expansions:

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{j=1}^{n-1} \left| m_j \right|^{\frac{j(n-j)}{2}}} \times W_J \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v_f, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}} \right),$$

$$g(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{B(m_1, \ldots, m_{n-1})}{\prod_{j=1}^{n-1} \left| m_j \right|^{\frac{j(n-j)}{2}}} \times W_J \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v_g, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}} \right).$$

Definition (Rankin-Selberg L-function)

Let $s \in \mathbb{C}$. Then the Rankin–Selberg L-function, denoted as $L_{f\times g}(s)$, is defined by

$$L_{f\times g}(s) = \zeta(ns) \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{A(m_1, \ldots, m_{n-1}) \cdot \overline{B(m_1, \ldots, m_{n-1})}}{(m_1^{n-1} m_2^{n-2} \ldots m_{n-1})^s},$$

which converges absolutely provided $\Re(s)$ is sufficiently large.

In the special case $g = f$, we have

$$L_{f\times f}(s) = \zeta(ns) \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{\left|A(m_1, \ldots, m_{n-1})\right|^2}{(m_1^{n-1} m_2^{n-2} \ldots m_{n-1})^s}$$

for $\Re(s) > 1$.

Euler Product

Fix $n \geq 2$. Let f, g be two Maass forms for $SL(n, \mathbb{Z})$ with Euler products

$$L_f(s) = \sum_{m=1}^{\infty} \frac{A(m, 1, \ldots, 1)}{m^s} = \prod_p \prod_{i=1}^{n} (1 - \alpha_{p,i} p^{-s})^{-1},$$

$$L_g(s) = \sum_{m=1}^{\infty} \frac{B(m, 1, \ldots, 1)}{m^s} = \prod_p \prod_{i=1}^{n} (1 - \beta_{p,i} p^{-s})^{-1},$$

then $L_{f \times g}(s)$ will have an Euler product of the form:

$$L_{f \times g}(s) = \prod_p \prod_{i=1}^{n} \prod_{j=1}^{n} (1 - \alpha_{p,i} \overline{\beta_{p,j}} p^{-s})^{-1}.$$

Functional Equation

For $n \geq 2$, let f, g be two Maass forms of types v_f, v_g for $SL(n, \mathbb{Z})$ whose associated L-functions L_f, L_g satisfy the functional equations:

$$\begin{aligned}\Lambda_f(s) &:= \prod_{i=1}^{n} \pi^{\frac{-s+\lambda_i(v_f)}{2}} \Gamma\left(\frac{s-\lambda_i(v_f)}{2}\right) L_f(s) \\ &= \Lambda_{\tilde{f}}(1-s), \\ \Lambda_g(s) &:= \prod_{j=1}^{n} \pi^{\frac{-s+\lambda_j(v_g)}{2}} \Gamma\left(\frac{s-\lambda_j(v_g)}{2}\right) L_g(s) \\ &= \Lambda_{\tilde{g}}(1-s),\end{aligned}$$

where $\tilde{f}, \tilde{g}$ are the Dual Maass forms.

The Rankin–Selberg L-function $L_{f\times g}(s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with at most a simple pole at $s = 1$ with residue proportional to $\langle f, g\rangle$, the Petersson inner product of f with g. $L_{f\times g}(s)$ satisfies the functional equation:

$$\begin{aligned}\Lambda_{f\times g}(s) &:= \prod_{i=1}^{n}\prod_{j=1}^{n} \pi^{\frac{-s+\lambda_i(v_f)+\overline{\lambda_j(v_g)}}{2}} \Gamma\left(\frac{s-\lambda_i(v_f)-\overline{\lambda_j(v_g)}}{2}\right) L_{f\times g}(s) \\ &= \Lambda_{\tilde{f}\times\tilde{g}}(1-s).\end{aligned}$$

We take $g = f$ and f to be a self-dual Maass form of type v so that

$$\begin{aligned}\Lambda_{f\times f}(s) &:= \pi^{\frac{-n^2 s}{2}} \prod_{i=1}^{n}\prod_{j=1}^{n} \Gamma\left(\frac{s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right) L_{f\times f}(s) \\ &= \Lambda_{f\times f}(1-s).\end{aligned}$$

- Let f be a self-dual Maass form for $SL(n, \mathbb{Z})$.
- We write $L_f(s)$ for the Godement–Jacquet L-function associated to f and $L_{f\times f}(s)$ for the Rankin–Selberg L-function of f with itself.
- Denote

$$L_{f\times f}(s) := \sum_{m=1}^{\infty} \frac{b(m)}{m^s}$$

which is absolutely convergent on $\Re(s) > 1$. The coefficients $b(m)$ are non negative and multiplicative.

- The inverse

$$\frac{1}{L_{f\times f}(s)} := \sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$

is also absolutely convergent on $\Re(s) > 1$.

- The coefficients $c(m)$ are multiplicative and can be written as

$$c(m) = \begin{cases} 0 & p^{n^2+1} | m \text{ for some } p \\ \prod_{p^l || m} (-1)^l \sum_{1 \le j_1 < \cdots < j_l \le n^2} \left| \alpha_{p,j_1} \ldots \alpha_{p,j_l} \right|^2 & \text{for all } l \le n^2. \end{cases}$$

For $\Re(s) > 1$,

$$\frac{1}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}.$$

The Mertens function is the sum

$$M(x) := \sum_{m \leq x} \mu(m).$$

Theorem 1 (von Mangoldt [11])

For sufficiently large x, $M(x) = o(x)$.

Theorem 2 (Davenport [1])

For any fixed $h \geq 0$,

$$M(x) \ll \frac{x}{(\log x)^h}.$$

[11] H. von Mangoldt, *Beweis der Gleichung $\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, Zweiter Halbband–Juli bis Dezember (1897), 835–852.

[1] H. Davenport, *On some infinite series involving arithmetical functions (II)*, The Quarterly Journal of Mathematics, **os-8(1)** (1937), 313–320. https://doi.org/10.1093/qmath/os-8.1.313

Theorem 3 (Landau [6])

For a positive constant c,

$$M(x) \ll x \exp(-c\sqrt{\log x}).$$

We define the analogue of Mertens function for $L_{f\times f}(s)$ as

$$\widetilde{M}(x) := \sum_{m\le x} c(m).$$

Theorem 4 (-, A. Sankaranarayanan)

For sufficiently large x and for some positive constant A, we have

$$\widetilde{M}(x) \ll_f x \exp(-A\sqrt{\log x}).$$

The Riemann Hypothesis for $L_{f\times f}(s)$ is equivalent to the estimate

$$\widetilde{M}(x) \ll_f x^{\frac{1}{2}+\epsilon} \quad \forall\, \epsilon > 0.$$

[6] E. Landau, *Beiträge zur analytischen Zahlentheorie*, Rendiconti del Circolo Matematico di Palermo (1884-1940), **26(1)** (1908), 169–302.

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Thank you