Elliptic fibrations & singularities to Anomalies & Spectra

F-theory = 12d perspective on type IIB theory.

"geometric - engineering"

→ a quantum theory of gravity which in some limit reduces to $SL(2,\mathbb{Z})$-equiv. type IIB supergravity.

IIB in 10d

$g_s = e^{\phi}$

- graviton $G_{\mu\nu}$
- scalar field ϕ
- 2-form field B_2

NS-NS sector

- scalar field C_0
- 2-form field C_2
- self-dual 4-form field C_4

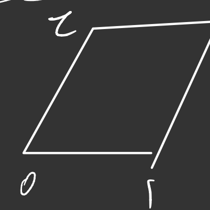
R-R sector

$* d\varphi = d\varphi$

+ fermionic

$SL(2,\mathbb{R})$ symmetry. ⟶ $SL(2,\mathbb{Z})$ "quantum effect"

{2 scalars, 2 2-form} each form 2d representations

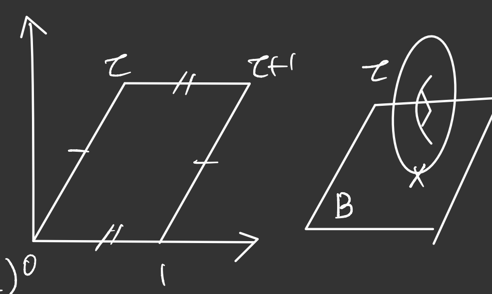
0	1	2	3	4	5	6	7	8	9	10	11
$\mathbb{R}^4$				Base ($B_3 \supset D_i$)						Elliptic Fiber	
4d spacetime				internal dimensions of 7-brane				transverse dimension onto 7-brane $(z, \bar{z})$		Axio-dilaton 	

Axio-dilaton $\quad \tau = C_0 + i e^{-\phi} = C_0 + \frac{i}{g_s}$

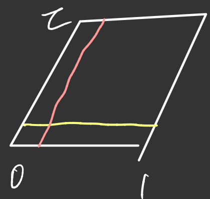
$\Updownarrow$ - S-duality $\qquad \tau \longrightarrow \frac{a\tau + b}{c\tau + d}$

the complex structure of the torus

$ad - bc = 1.$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$



M thry on $T^2 \longrightarrow$ IIB



M2 brane on a-cycle = F1 string

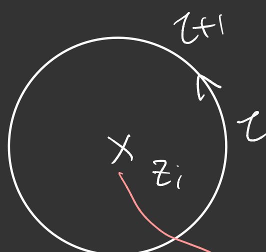
M2 brane on c-cycle = D1 string.

M2 brane $pa + qb$-cycle

$\Rightarrow$ (p, q) string.

D7 brane = magnetic source for C_0.

$z, \bar{z}$

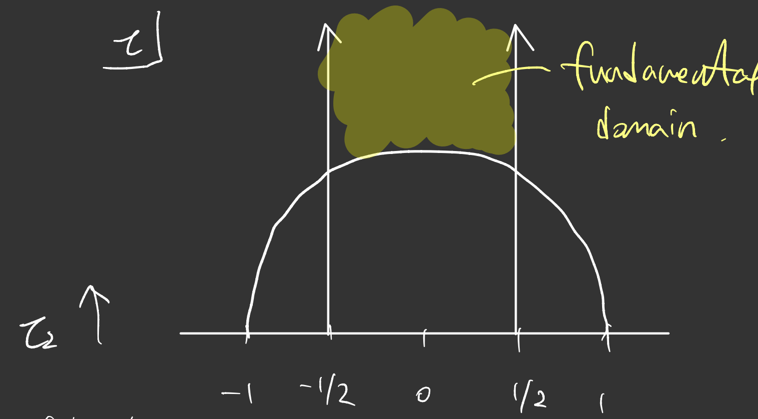


$$\tau \sim \frac{i}{g_s} + \underbrace{\frac{1}{2\pi i} \log(z - z_i)}$$

$z \to z_i$, $\tau \to i\infty$, $e^{\phi} (= g_s) \longrightarrow 0$.

D7 brane the location itself = "a weakly-coupled region"

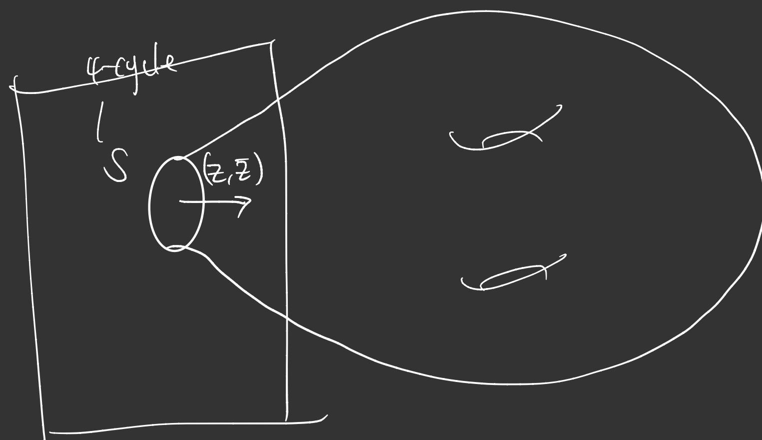
$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad \underline{\underline{SL(2,\mathbb{Z})}}$$



Modular inv. function $j(\tau(z))$

$$\frac{P(z)}{Q(z)}$$

→ elliptic fibration inv.



A Noetherian scheme X

An algebraic cycle of X $\Rightarrow$ $\sum_i N_i V_i$ subvarieties

integer coeff.

If $\dim(V_i) = d \ \forall i$, "$d$-cycle".

The group of all cycles $Z(X) = \bigoplus_d Z_d(X)$

the free group generated by subvarieties of dim d.

The degree of a zero-cycle $\sum_i N_i p_i$

$$\deg\left(\sum_i N_i p_i\right) = \sum_i N_i [\kappa(p_i) : k].$$

Consider θ an algebraic one-cycle. $\theta = \sum_i m_i \theta_i$.

Denote $\theta_i \cdot \theta_j$ $(i \neq j)$ the zero-cycle defined by the intersection of θ_i and θ_j. $(i \neq j)$.

ex)
Intersect $x-y=0$ and $x^2+y^2+z^2=0$.

$[0;1;\sqrt{2}]$ and $[0;-1;\sqrt{2}]$ $\Rightarrow$ zero cycle.

formal sum of this $\Rightarrow$ the intersection

An "n-point" of θ = a point $\bigcup_i \theta_i$

If it does NOT have n-pts for $n \geq 2$, θ is a "tree".

Let's say we have 2 curves. intersect transversally.

$\Rightarrow$ their intersection consists of isolated reduced closed points.

Fiber Types (a la Kodaira)

The types of $\Theta \in Z_1(X)$ consists of the isomorphism class of each irred. curve Θ_i w/ the topological structure of the reduced polyhedron $\sum_i \Theta_i$.

↳ characterized by the underlying set of the $\Theta_i \cap \Theta_j$ $(i \neq j)$.

Dual graph of $\Theta = \sum_i m_i \Theta_i$ = an associated graph to Θ s.t.

1) the vertices are the irred. components (Θ_i) of the fiber
2) the weight of the vertex corr. to an irred. comp. (Θ_i) is its mult. (m_i).
3) the vertices corr. to the Θ_i and Θ_j are connected by $\hat{\Theta}_{ij} = \deg(\Theta_i \cap \Theta_j)$ edges.

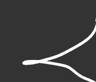
Kodaira type = the type of a geometric fiber over a codim-1 pt of a minimal elliptic fibration.

Kodaira '63 & Néron '64 $\Rightarrow$ 10 types

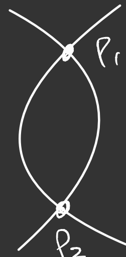
Kodaira	I_0	$I_{n>0}$	II	III	IV	I_0^*	I_n^*	IV^*	III^*	II^*
Néron	A	B_n	C_1	C_2	C_3	C_4	$C_{5,n}$	C_6	C_7	C_8

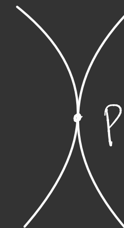
I_0 : a smooth curve of genus 1.

I_1 : an irred. nodal rational curve  dual graph

II : an irred. cusp. rational curve  $\tilde{A}_0$

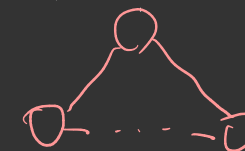
I_2 : $\theta = \theta_1 + \theta_2, \quad \theta_1 \cdot \theta_2 = p_1 + p_2$





dual graph

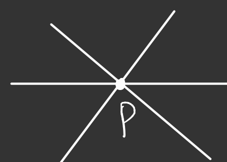
$\tilde{A}_1$

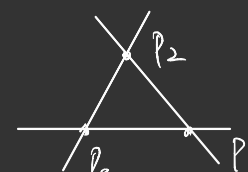


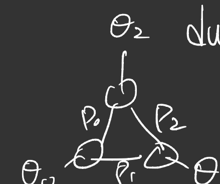
dual graph $\tilde{A}_{n-1}$

III : $\theta = \theta_1 + \theta_2, \quad \theta_1 \cdot \theta_2 = 2p.$

IV : $\theta = \theta_1 + \theta_2 + \theta_3, \quad \theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_3 = \theta_1 \cdot \theta_3 = p.$







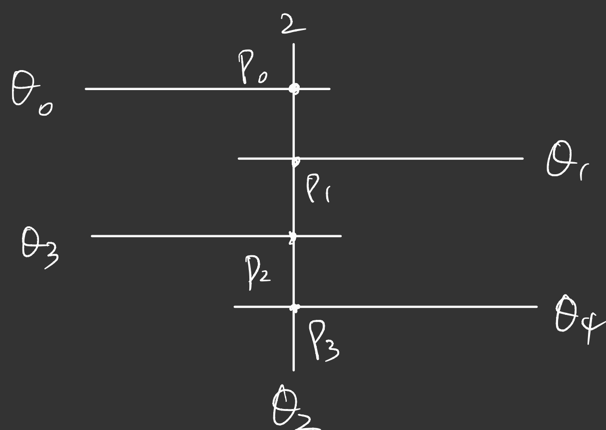
dual graph $\tilde{A}_2$

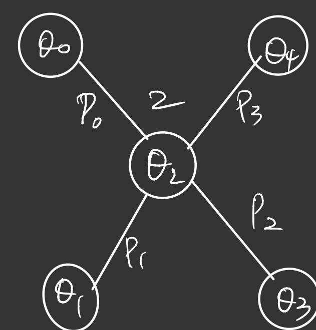
I_n $(n \geq 3)$: $\theta = \theta_0 + \cdots + \theta_{n-1}, \quad \theta_i \cdot \theta_{i+1} = p_i, \quad \theta_{n-1} \cdot \theta_0 = p_0.$

cf) $n=3$: $\theta = \theta_0 + \theta_1 + \theta_2, \quad \theta_0 \cdot \theta_1 = p_1, \quad \theta_1 \cdot \theta_2 = p_2, \quad \theta_2 \cdot \theta_0 = p_0.$

Type I_0^*: $\theta = \theta_0 + \theta_1 + \underline{2\theta_2} + \theta_3 + \theta_4$

$\theta_0 \cdot \theta_2 = p_0, \quad \theta_1 \cdot \theta_2 = p_1, \quad \theta_2 \cdot \theta_3 = p_2, \quad \theta_4 \cdot \theta_2 = p_3$





dual graph

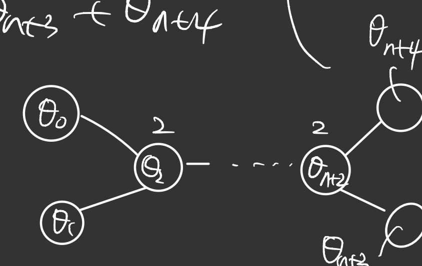
$\tilde{D}_4$

$\tilde{D}_{4+n}$

Type I_n^*: $\theta = \theta_0 + \theta_1 + 2\theta_2 + \cdots + 2\theta_{n+2} + \theta_{n+3} + \theta_{n+4}$

for $i = 1, \ldots, n+2$ $\quad \theta_i \cdot \theta_{i+1} = p_i$

$\theta_0 \cdot \theta_2 = p_0, \quad \theta_{n+4} \cdot \theta_{n+2} = p_{n+3}$

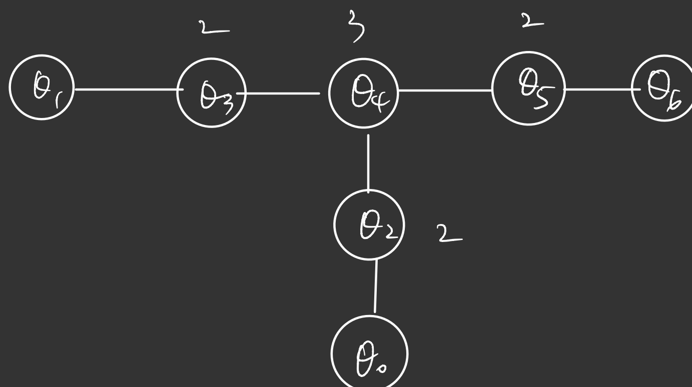


Type IV*:

$$\Theta = \Theta_0 + \Theta_1 + 2\Theta_2 + 2\Theta_3 + 3\Theta_4 + 2\Theta_5 + \Theta_6$$

$$\Theta_i \cdot \Theta_{i+1} = p_i \qquad i = 3, 4, 5$$

$$\Theta_0 \cdot \Theta_2 = p_0, \qquad \Theta_1 \cdot \Theta_3 = p_1, \qquad \Theta_2 \cdot \Theta_4 = p_2$$



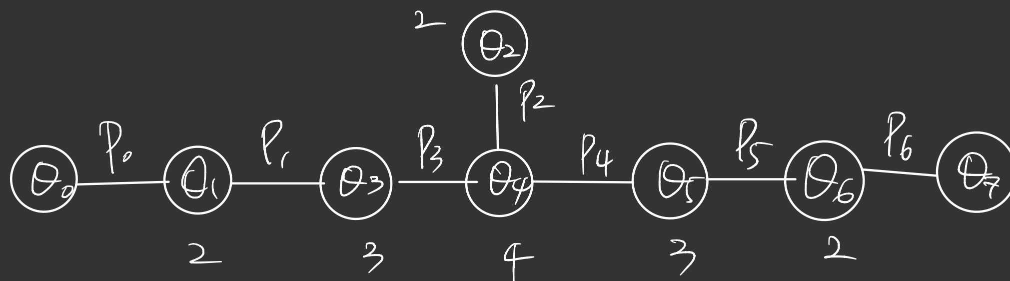
dual graph $\tilde{E}_6$.

III^*:

$$\Theta = \Theta_0 + 2\Theta_1 + 2\Theta_2 + 3\Theta_3 + 4\Theta_4 + 3\Theta_5 + 2\Theta_6 + \Theta_7$$

$$\Theta_i \cdot \Theta_{i+1} = p_i \quad \text{for } i = 3, 4, 5, 6$$

$$\Theta_0 \cdot \Theta_1 = p_0, \quad \Theta_1 \cdot \Theta_3 = p_1, \quad \Theta_2 \cdot \Theta_4 = p_2$$

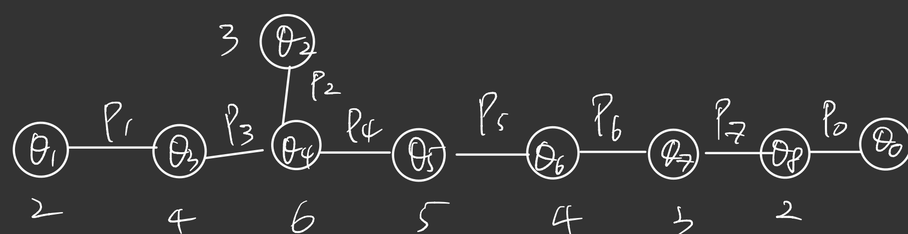


dual graph $\simeq \tilde{E}_7$

II^*: $\theta = \theta_0 + 2\theta_1 + 3\theta_2 + 4\theta_3 + 6\theta_4 + 5\theta_5 + 4\theta_6 + 3\theta_7 + 2\theta_8$

$\theta_i \cdot \theta_{i+1} = p_i \quad (i = 3, \cdots, 7)$

$\theta_0 \cdot \theta_8 = p_8, \quad \theta_1 \cdot \theta_3 = p_1, \quad \theta_2 \cdot \theta_4 = p_2$



dual graph

$\tilde{E}_8$

G: a simply-connected simple Lie algebra w/ $\underline{\mathfrak{g}}$.

$\tilde{\mathfrak{g}}$: the affine Dynkin diagram that reduces upon removal of its extra nodes to $\mathfrak{g}$.

$\tilde{\mathfrak{g}}^t$: its Langlands dual

$\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^t$ are distinct only when $\underline{\mathfrak{g}}$ is not simply-conn.

BCFG type.

Elliptic fibration = a surjective proper morphism $\varphi : \underline{Y} \to \underline{B}$
if the generic fiber of φ is a smooth proj. curve of genus 1 and φ has a rational section.

B is a curve $\Rightarrow$ Y is an elliptic surface

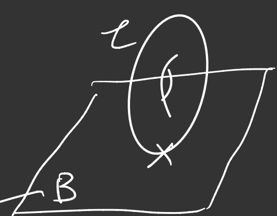
B is a surface $\Rightarrow$ Y is an elliptic 3-fold

B is of dim $n-1$ $\Rightarrow$ Y is an elliptic n-fold.

The locus of singular fibers of φ

= the discriminant locus $\Delta(\varphi)$ $\Rightarrow$ divisor.

smooth.



$\varphi : Y \to B$ a morphism of schemes.

For any $p \in B$, the fiber over p is $Y_p = Y \times_B \operatorname{Spec} \kappa(p)$.

The 1st proj. $Y_p \to Y$ induces an homeomorphism from Y_p onto $\varphi^{-1}(p)$

The 2nd proj. gives Y_p the structure of a scheme over the residue field $\kappa(p)$

p is not a closed pt $\Rightarrow$ $\kappa(p)$ is not necessarily alg. closed.

$\to$ certain comp. of Y_p can be $\kappa(p)$-irred. while they become reducible after an appropriate field extension.

An irred. scheme over a field k is geometrically irred. when it stays irred. over ANY field extension.

The most refined description for Y_p is always the one corr. to $\overline{\kappa(p)}$.

$\Rightarrow$ the geometric fiber over p is the fiber $\underline{\underline{Y_p \times_{\kappa(p)} \overline{\kappa(p)}}}$

$\Rightarrow$ A geometric fiber is always composed of geometrically irred. comp.

$\Rightarrow$ The type of Y_p is geometric if it does not change under a field extension.

"arithmetic fiber"

The Kodaira fibers of type I_1, II, III, III^*, II^* never need a field ext.

the rest IV, $I_{n\geq 1}$, I_n^*, IV^* can come from fibers Y_p whose types are not geometric & require at least a field ext. of deg 2 to describe a fiber w/ a geometric type.

→ Y_p has a geometric type : split. IV^s, $I^s_{n\geq 2}$, $I^{*s}_{n\geq 1}$

→ otherwise : nonsplit. IV^{ns}, $I^{ns}_{n\geq 2}$, $I^{*ns}_{n\geq 1}$

I_0^*

split

x field ext.

$\rightsquigarrow \tilde{D}_4$

semi-split

a quadratic ext.

$\downarrow$

$\tilde{B}_3^t$

nonsplit.

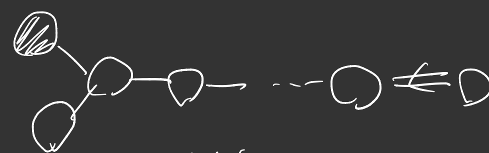
a cubic ext.

$\rightsquigarrow \tilde{G}_2^t$

$\tilde{F}_4^t$ from IV^{*ns}

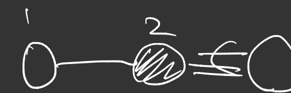
$\tilde{C}_{2+\ell}^t$ $(\ell \geq 0)$ from $I_{4+2\ell}^{ns}$

$I_{5+2\ell}^{ns}$



$\tilde{B}_{3+\ell}^t$ $(\ell \geq 0)$

$I_0^{*ss} \to I_\ell^{*ns}$



$\tilde{G}_2^t$

I_0^{*ns}

Elliptic fibrations & singularities to Anomalies & Spectra (Lecture 2)

elliptic fibration $\varphi : Y \to B$

A genus-one fibration over a variety B
= a surj. morphism $\varphi : Y \to B$ onto B s.t. the general fiber is smooth proj. curve of genus-one.

A rational section of the genus 1 fibration
= a rational map $\sigma : B \longrightarrow Y$ s.t. the image of $\varphi \circ \sigma$ is dense in B and restrict to the id on the domain of σ.

$\Rightarrow$ When the genus-1 fibration admits a rational section, we call it an elliptic fibration.

$\longrightarrow$ birational to Weierstrass model.

Weierstrass Model

$\mathcal{L}$ a line bundle over a quasi-proj. var. B

the proj. bundle $\pi : X_0 = \mathbb{P}_B[\underline{\mathcal{O}_B} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}] \longrightarrow B$

a section of $\mathcal{O}_{X_0}(1)$

a section of $\mathcal{O}_{X_0}(1) \otimes \pi^* \mathcal{L}^{\otimes 2}$

a section of $\mathcal{O}_{X_0}(1) \otimes \pi^* \mathcal{L}^{\otimes 3}$

$$\begin{array}{ccc}
[\, z, & x, & y \,] \\
\rotatebox[origin=c]{-90}{$\in$} & \rotatebox[origin=c]{-90}{$\in$} & \rotatebox[origin=c]{-90}{$\in$} \\
\mathcal{O}_B & \mathcal{L}^{\otimes 2} & \mathcal{L}^{\otimes 3} \\
\downarrow & \downarrow & \downarrow \\
\mathcal{O}_B \oplus & \mathcal{L}^{\otimes 2} & \oplus\, \mathcal{L}^{\otimes 3}
\end{array}$$

$\underline{\mathcal{O}(3) \otimes \pi^* \mathcal{L}^{\otimes 6}}$

the most general Weierstrass eq'n.

$$0 = y^2 z + \underline{a_1} xyz + \underline{a_3} yz^2 - (x^3 + \underline{a_2} x^2 z + \underline{a_4} xz^2 + \underline{a_6} z^3).$$

a_i is a section of $\pi^* \mathcal{L}^{\otimes i}$ on B.

w/ a rational pt $(x = z = 0)$.

$\varphi : Y \to B$.

$\mathcal{L}$: fund. line bundle of the Weierstrass model.

→ can be direcly defined from the elliptic fibration Y as

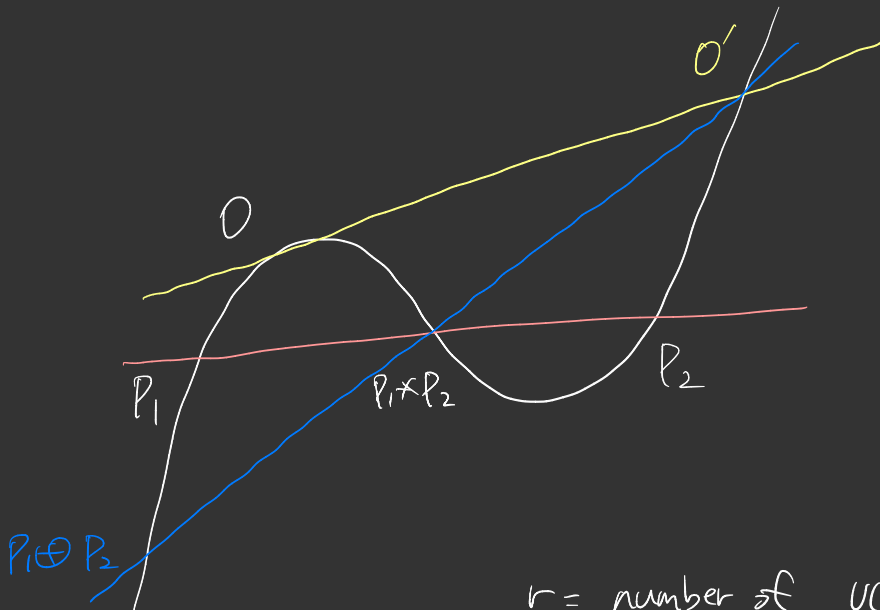
$$\mathcal{L} = R^1 \varphi_* \mathcal{O}_Y .$$

The rational section $\underbrace{O: z = x = 0}$.

Mordell-Weil group = the group of rational sections of $\varphi : Y \to B$.

→ a finitely generated Abelian group. → its ranks & tension groups are birational inv. of the elliptic fibration.

→ ∃ an id: for a Weierstrass model, $O: z = x = 0$.



$$P_1 \oplus P_2 = (P_1 * P_2) * O'$$

$$\mathbb{Z}^{r} \oplus (\text{Torsion})$$

rank

$r =$ number of $U(1)$'s

$$0 = y^2 z + a_1 xyz + a_3 yz^2 - (x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3)$$

Tate and Deligne '72

$$b_2 = a_1^2 + 4a_2$$
$$b_4 = a_1 a_3 + 2a_4$$
$$b_6 = a_3^2 + 4a_6$$
$$b_8 = a_1^2 a_6 - a_1 a_3 a_4 + 4a_2 a_6 + a_2 a_3^2 - a_4^2$$
$$c_4 = b_2^2 - 24 b_4$$
$$c_6 = -b_2^3 + 36 b_2 b_4 - 216 b_6$$
$$\Delta = -b_2^2 b_8 - 8 b_4^3 - 27 b_6^2 + 9 b_2 b_4 b_6 \quad \leftarrow \text{ a section of } \pi^* \mathcal{L}^{\otimes 12}$$
$$j = c_4^3 / \Delta$$

Δ → the locus of pts p of B s.t. the fiber over p (Y_p) is singular.

two birational transform on $\mathbb{P}^2$

$$[x, y, z] \longmapsto [x, \tfrac{1}{2}(y - a_1 x - a_3 z), z]$$
$$[x, y, z] \longmapsto [x - 3b_2 z, \tfrac{1}{3} y, 36 z]$$
$$\Rightarrow y^2 z = x^3 - \underbrace{27 c_4}_{= f} xz^2 - \overbrace{54 c_6}^{= g} z^3$$

"the short form"

$$1728 \Delta = c_4^3 - c_6^2$$
$$4 b_8 = b_2 b_6 - b_4^2.$$

b_i, c_i are sections of $\pi^* \mathcal{L}^{\otimes i}$

$$y^2 z = x^3 + fxz^2 + gz^3 . \quad \text{(the short form)}$$

$f = -27c_4$

$g = -54c_6$.

$z=1$ / $\quad [x/z, y/z, 1] \longmapsto [x, y]$.

$$\underline{y^2 = x^3 + fx + g} . \quad \text{in } \mathbb{P}^2 \setminus \{z=0\} \cong \mathbb{A}^2 .$$

$$\Delta = \frac{c_4^3 - c_6^2}{1728} \sim 4f^3 + 27g^2$$

7-branes live here!

$$j = \frac{c_4^3}{\Delta} = 1728 \frac{4f^3}{4f^3 + 27g^2} \sim 1728 \frac{4f^3}{\Delta} .$$

characterizes smooth fibers upto isomorphism.

"Tate's algorithm"

Tate's algorithm

R a complete discrete valuation ring w/ valuation v
uniformizing parameter s

& perfect residue field $k = R/(s)$.
characteristic 0.

has only 3 ideals
1. 0 2. ring itself 3. sR.

→ The scheme $\mathrm{Spec}(R)$ has only 2 pts:
1. the generic pt (defined by the 0 ideal)
2. the closest pt (" the principal ideal sR).

E/R an elliptic curve over R w/ Weierstrass eq'n.

→ The generic fiber is a regular elliptic curve.

→ Do a resolution of singularities ⟹ a regular model $\mathcal{E}$ over R.

& special fiber = the fiber over the closed pt $\operatorname{Spec} R/(s)$.

Tate's algorithm:

the type of the geometric fiber over the closed pt of $\operatorname{Spec} R$

by manipulating the valuations of the coeff & Δ.

& the arithmetic properties of some aux. poly.

becomes geometric by a field extension $\mathcal{K}'/\mathcal{K}$.

"local index"

= the order of the component group

= # of reduced comp. of the special fiber defined over $\mathcal{K}$.

Tate's notation: $a_{i,j} = a_i s^{-j}$. $a = a_{i,j} s^j$ $\begin{cases} f = f_j s^j \\ g = g_k s^k \end{cases}$

Tate step	K. Type	$v(f)$	$v(g)$	$v(\Delta)$	$v(j)$	j	Monodromy	Dual graph.
1	I_0	≥ 0	≥ 0	0	0	$\mathbb{C}$	I_2	–
2	I_1	0	0	1	-1	∞	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	 $\tilde{A}_0$
	I_n	0	0	$n>1$	$-n$	∞	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	$\tilde{A}_{n-1}$
3	II	≥ 1	1	2	0	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	 $\tilde{A}_0$
4	III	1	≥ 2	3	0	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	 $\tilde{A}_1$
5	IV	≥ 2	2	4	0	0	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	 $\tilde{A}_2$
6	I_0^*	2 ≥ 2	≥ 3 3	6 6	0	$\mathbb{C}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\tilde{D}_4$
7	I_n^*	2 ≥ 2	≥ 3 3	$n+6$	$-n$	∞	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	$\tilde{D}_{n+4}$
8	IV^*	≥ 3	4	8	0	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\tilde{E}_6$
9	III^*	3	≥ 5	9	0	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\tilde{E}_7$
10	II^*	≥ 4	5	10	0	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\tilde{E}_8$

Engineer the axio-dilaton field (τ) on space B

Captures nonperturbative aspect :

τ is magnetic dual to the 7-brane .

Detect 7-brane by its monodromy

nontrivial only if the fiber is singular .

classified by Kodaira classification

Can be deduced w/ Tate's algorithm $\Rightarrow$ ADE Dynkin graph.

$\Rightarrow$ Associate ADE algebra

$\rightarrow$ ADE classifications

$\rightarrow$ get exceptional gauge groups .

" G- models "

fiber type

$\boxed{K}$ - model

a smooth divisor

$\Delta(\varphi)$ contains only irred comp S s.t. S is of type K.

(∟ any fiber away from S irred.)

Cartier divisor $\Delta \rightarrow$ irred comp. Δ_i

$\Rightarrow \mathfrak{g}_i$ is the trivial Lie algebra ($\therefore \mathfrak{g}_i^t = \tilde{A}_0$)

Lie algebra associated w/ φ : $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$

A Lie group G attached to a given φ depends on

1) the type of generic sing. fibers

2) the $MW(\varphi)$.

trivial $MW(\varphi)$; G is semisimple : $G = e^{\mathfrak{g} = \oplus_i \mathfrak{g}_i}$

ex) (K_1+K_2)-model $\tilde{\mathfrak{g}}_1^t$, $\tilde{\mathfrak{g}}_2^t$ $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $G = e^{\mathfrak{g}}$.

$MW(\varphi) = T \times \mathbb{Z}^r$

$$G = \underbrace{\tilde{G}/T}_{\text{requires a choice of embedding } T \text{ into the center } Z(\tilde{G}) \text{ of } G} \times U(1)^r, \qquad \tilde{G} = e^{\mathfrak{g} = \oplus_i \mathfrak{g}_i}.$$

i.e. $Z(G) = Z(\tilde{G})/T$

Lie alg. $\mathfrak{g}$ associated φ is the Langlands dual $\mathfrak{g}^\vee = \oplus_i \mathfrak{g}_i$ of $\mathfrak{f} = \oplus_i \mathfrak{f}_i$

Denote $e^{\mathfrak{g}^\vee}$ the unique simply-conn simple group w/ $\mathfrak{g}^\vee$

$$G = \frac{e^{\mathfrak{g}^\vee}}{MW_{tor}(\varphi)} \times U(1)^{\operatorname{rk} MW(\varphi)}.$$

<u>G-model</u> = an elliptic fibration $\varphi : Y \to B$ w/ Δ containing an irred. comp. S s.t.

1. the generic fiber over any other component of Δ is irred.
2. the fiber over the generic pt of S has a dual graph that becomes of the same type as the Dynkin diagram of the Langlands dual of $\mathfrak{g}$ after removing the node corr. to the component touching the section of φ.

G	K
A_0	I_1, II
A_1	I_2^{s}, I_2^{ns}, I_3^{ns}, III, IV^{ns}
$A_{n\geq 2}$	I_n^{s}
D_{n+4}	I_n^{*s}
E_6	IV^{*s}
E_7	III^{*}
E_8	II^{*}
B_3	I_0^{*ss}
$B_{n+3\geq 4}$	I_n^{*ns}
C_{n+2}	I_{2n+4}^{ns}, I_{2n+5}^{ns}
F_4	IV^{*ns}
G_2	I_0^{*ns}

$S_i > 1$

Elliptic fibrations & singularities to Anamalies & Spectra (Lecture 3)

$\varphi : Y \to B$ a smooth flat fibration.

Δ has a unique component S over which the generic fiber is reducible w/ dual graph $\tilde{g}^{t}$.

irred. comp: $\underset{\text{zero node}}{D_0}, \cdots, D_n$

C_a: the irred. comp. of the generic over S.

$\overline{\omega}_S(C) = (\underbrace{-D_1 \cdot C}, \cdots, \underbrace{-D_n \cdot C})$

intersection #s

$$\varphi^*(S) = \sum_a m_a \underbrace{D_a}$$

fibral divisor

$\to$ C_a is the generic fiber of D_a over S.

The irred. curves of the degenerations over codim 2 loci

$\Downarrow$

give weights of R.

The weight of a vertical C w.r.t. D_a

= the intersection #

$$\overline{\omega}_a(C) = -\int_Y D_a \cdot C$$

$$\overline{\omega}(C) = \boxed{\overline{\omega}_1(C) \cdots\cdots \overline{\omega}_n(C)}$$

A saturated set of weights is inv. under the action of the Weyl group.

A set Π of integral weights is saturated if for any weight $\varpi \in \Pi$ and any simple α, the weight $\varpi - i\alpha \in \Pi$ for any $0 \le i \le \langle \varpi, \alpha^\vee \rangle$.

A saturation of a subset of weights $\bar{\Pi}$ is finite iff Π is finite

Any subset Π of weights is contained in a unique smallest saturated subset $\Rightarrow$ the saturation of Π.

A saturated set w/ highest weight λ consists of all dominant weights lower than or equal to λ & their conjugates under the Weyl group.

if λ is a dominant weight and $\mu \prec \lambda$ $\forall \mu \in \Pi$.

$\Rightarrow$ w/ Π a finite saturated set of weights,

$\exists$ a finite-dim $\mathfrak{g}$-module whose set of weights is Π.

To a G-model:

the weight vectors of the irred. vert. rational curves of the fibers over codim 2 pts form a set Π whose saturation defines uniquely a R.

Aspinwall - Gross '96.

Dictionary

elliptic fibration	gauge theory
codim 1 singularities	gauge algebra ($\mathfrak{g}$)
codim 2 singularities	representation (R)
Mordell-Weil group	the fund. group of the gauge group ($\pi_1(G)$).
crepant resolutions	Coulomb phases
flops	phase transitions
triple intersection polynomial	5d prepotential

F-theory compactifications

$$\begin{tikzcd}
\text{M theory} \arrow[r] \arrow[d, "CY_3"'] & \text{F theory} \arrow[d, "CY_3"] \\
\text{5d } \mathcal{N}=1 \text{ theory} \arrow[r] & \text{6d } \mathcal{N}=(1,0) \text{ theory}
\end{tikzcd}$$

gravity mult

$g_{\mu\nu}, \underline{\psi_\mu}, A_\mu$

vector mult

$A_\mu, \underline{\lambda}, \phi$

hypers

$\xi^m, \underline{A}^m$

gravity mult $g_{\mu\nu}, B^-_{\mu\nu}, \underline{\psi^A_\mu}$

tensor mult $B^+_{\mu\nu}, \phi, \underline{\chi^{A+}}$

vector mult $A, \underline{\lambda^{A-}}$

hypers q, ψ^+

tensor mult : $SO(1, n_T)/SO(n_T)$

hyper : n_H

Coulomb branch of the 5d theory.

= the vacuum moduli space where the $\mathfrak{g}$ is completely broken into Cartan subalg $\mathfrak{h}$ by the vev of the ϕ in vector mult.

residual gauge sym. = Weyl group $\mathcal{W}$ of $\mathfrak{g}$.

Full gauge-fixing = A choice of a dual fund. Weyl chamber $\mathcal{W}_0$

$$\mathcal{W}_0 = \{\phi \in \mathfrak{h} : \underline{\alpha_i}(\phi) > 0,\ i = 1, \cdots, \underline{r}\}$$

mass of a hyper $\propto$ its charge

given by λ of R.

$\lambda^\perp \to$ massless

λ: $\ker(\lambda) \subset \mathcal{W}_0 \subset \mathfrak{h}$

$$\begin{cases} \text{vector mult} \longrightarrow \text{Weyl chamber} \\ \text{massless hyper @ the singularities} \longrightarrow \text{subchamber structures.} \end{cases}$$

The network of flops via hyperplane arrangement

$\Rightarrow$ the theory of Coulomb branches of 5d gauge theories ($\mathcal{N}=1$).

The network of crepant resolutions

$\cong$ the network of Weyl chambers of hyperplane geometry $I(\mathfrak{g}, R)$

A resolution of singularities of a variety Y

= a proper birational morphism $\varphi : \tilde{Y} \to Y$

($\tilde{Y}$ is nonsingular)

& φ is an isomorphism away from the singular locus of Y.

i.e. $\tilde{Y}$ is nonsingular

if U is the sing. locus of Y, φ maps to $\varphi^{-1}(Y \setminus U)$

isomorphically onto $Y \setminus U$,

A birational proj. morphism φ is crepant if it preserves the canonical class.

i.e. $K_{\tilde{Y}} = \varphi^* K_X$.

In $d=2$, one unique crepant resolution

In $d=3$, crepant resolutions of Gorenstein singularities always exist
but usually not unique.

In $d \geq 4$, crepant resolutions are not always possible.

Coulomb branch = network of resolutions

subchambers = crepant resolutions

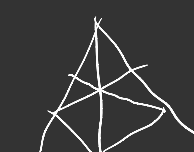
walls & their intersections = partial resolutions

Moving on the walls & their intersections = blowing down.

reflections = flops.

ex) $SU(3)$ 

$SU(2) \times G_2$ ('18) 

$SU(2) \times SU(3)$ ('19) 

5d/6d Spectra

F theory on CY_3 $\downarrow$ 6d (1,0) supergravity	M theory on CY_3 $\downarrow$ 5d $\mathcal{N}=1$ supergravity	F theory on $CY_3 \times S^1$ $\downarrow$ 5d $\mathcal{N}=1$ supergravity.

$$n_V^{(6)} = h^{1,1}(CY_3) - h^{1,1}(B) - 1$$

$$n_T = h^{1,1}(B) - 1$$

$$n_H = \underline{n_H^{0}} + n_H^{ch}$$

$$n_H^{0} = h^{2,1}(CY_3) + 1$$

$$n_V^{(5)} = n_V^{(6)} + n_T + 1 = h^{1,1}(CY_3) - 1$$

$$n_H^{0} = h^{2,1}(Y) + 1$$

$$h^{0,1}(B) = h^{0,2}(B) = 0$$

canonical class of the CY_3

$$10 - K^2 \text{ (Noether)}$$

Shioda-Tate-Wazir thm $({}^{1}1(Y))$

$$\begin{cases} h^{1,1}(Y) = \boxed{h^{1,1}(B)} + \boxed{f} + 1, \quad f = \mathrm{rank}(G). \\ h^{2,1}(Y) = h^{1,1}(Y) - \frac{1}{2}\chi(Y) \end{cases}$$

Intersection theory & Pushforward formulae.

$\chi(X) = \int_X c(X)$

The chow group $A_*(X)$ = the group of divisors

$[V]$ = the class of a subvariety V in $A_*(X)$.

$\alpha \in A_*(A)$, $\int_X \alpha$ = degree of α.

$\forall \alpha \in A_*(X)$, $\int_X \alpha = \int_Y f_*\alpha$.

The total homological Chern class $c(X) = c(TX) \cap [X]$

$c_i(TX)$; ith chern class of the TX

$f: X \to Y$ f_* pushforward

$$f^*[V] = [f^{-1}(V)]$$

image $W = f(V)$ a subvariety of Y, a fun field $R(V)$ an extention of the $R(W)$.

$$f_*: A_*(X) \to A_*(Y) : \quad f_*[V] = \begin{cases} 0 & \dim V \neq \dim W \\ [R(V):R(W)]\,[W] & \text{if } \dim V = \dim W \end{cases}$$

X a proj. var. w/ at most Gorenstein sing.

Y a crepant resolution w/ $K_Y = f^* K_X$.

Aluffi '10 $\quad Z \subset X$ a complete intersection of hypersurfaces $Z_i = V(z_i)$ meeting transversally in X.

$i = 1, \cdots, d$.

$f : \tilde{X} \to X$ centered @ Z.

$$X \xleftarrow{(z_1, \cdots, z_n \mid e)} \tilde{X} = Bl_Z X.$$

exceptional divisor $E = V(e)$.

The total Chern class $c(T\tilde{X}) = (1+E)\left(\prod_{i=1}^{d} \frac{1+f^*Z_i - E}{1+f^*Z_i}\right) f^* c(TX)$

$f_* E^n = (-1)^{d+1} \underline{h_{n-d}(Z_1, \cdots, Z_d)} Z_1 \cdots Z_d$.

'17

$$\tilde{Q}(t) = \sum_a f^* Q_a t^a \qquad Q(t) = \sum_a Q_a t^a$$

$$f_* \tilde{Q}(t) = \sum_{\ell=1}^{d} Q(z_\ell) M_\ell, \qquad M_\ell = \prod_{\substack{m=1 \\ m\neq \ell}}^{d} \frac{z_m}{z_m - z_\ell}.$$

$$\pi : X_0 = \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}] \to B.$$

$$c_1(\mathcal{O}_{X_0}(1)) = H$$
$$c_1(\mathcal{L}) = L$$

$$\pi_* \tilde{Q}(H) = -\frac{Q(H)}{H^2}\Big|_{H=-2L} + 3\frac{Q(H)}{H^2}\Big|_{H=-3L} + \frac{Q(0)}{6L^2}$$

⇒ the chern class of a Weierstrass model

$$c(TY) = \frac{(1+H)(1+H+2L)(1+H+3L)}{1+3H+6L}\, c(TB).$$

ex) two divisors of class Z_1 and Z_2

$$f_* E = 0 \qquad f_* E^2 = -Z_1 Z_2 \qquad f_* E^3 = -(Z_1+Z_2) Z_1 Z_2 \quad \cdots$$

ex) 3 divisors

$$f_* E = 0 \qquad f_* E^2 = 0 \qquad f_* E^3 = Z_1 Z_2 Z_3 \qquad f_* E^4 = (Z_1+Z_2) Z_1 Z_2 Z_3 \quad \cdots$$

for Weierstrass proj. bundle π

$$\pi_* 1 = 0 \qquad \pi_* H = 0 \qquad \pi_* H^2 = 1 \qquad \pi_* H^3 = -5L \quad \cdots$$

$$\pi_* H^k = \left[(-2)^{k-1} - (-3)^{k-1} \right] L^{k-2}$$

$$\Rightarrow \pi_* \left(H^k \, \underline{(3H+6L)} \right) = -(-3)^k L^{k-1}$$

Elliptic fibrations & singularities to Anomalies & Spectra (Lecture 4)

Batyrev: The Euler characteristic of a crepant resolution of a variety w/ Gorenstein canonical singularities is independent on the choice a crepant resolution.

→ Identify χ as the degree of the total homological Chern class of a crepant resolution $f : \tilde{Y} \to Y$ for a Weierstrass model $\varphi : Y \to B$ is

$$\chi(\tilde{Y}) = \int c(\tilde{Y}) = \int_B \pi_* f_* c(\tilde{Y})$$

$$X_0 = [\mathcal{O}_B \oplus \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 3}]$$

→ $\underbrace{c(B),\ c(\mathscr{L}),\ S}_{}$

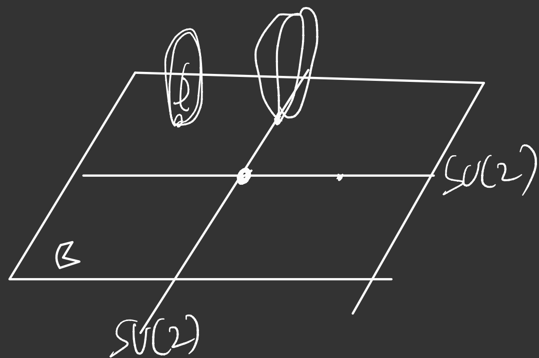
Kontsevich: (for CY_3) Hodge #s are the same for two different crepant resolutions.

ex) smooth Weierstrass model

- generating fcns of χ: $\chi(Y) = \frac{12L}{1+6L} c(B)$, $\quad L = -K$.
- $\chi(CY_3) = -60K^2$.
- Hodge #'s: $h^{1,1}(CY_3) = 11 - K^2$, $\quad h^{2,1}(CY_3) = 11 + 29K^2$.

ex) G_2 Weierstrass Model

- generating fcns of χ: $\chi(Y) = 12L \frac{L + 2SL - S^2}{(1+S)(1+6L-3S)} c(B)$.
- $\chi(CY_3) = -60K^2$
- Hodge #'s: $h^{1,1}(CY_3) = 13 - K^2$, $h^{2,1}(CY_3) = 13 + 29K^2 + 24K \cdot S + 2S^2$.



ex)

$$SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2 .$$

$$\chi(CY_3) = -4(9K^2 + 4K\cdot S + S^2)$$

$$h^{1,1} = 13 - K^2$$

$$h^{2,1} = 13 + 17K^2 + 8K\cdot S + 2S^2 .$$

Fourfolds ? Chern numbers

$$\int_Y c_1(TY)^4, \quad \int_Y c_1(TY)^2 c_2(TY), \quad \int_Y c_1(TY) c_3(TY),$$

$$\int_Y c_2^2(TY), \quad \int c_4(TY) .$$

Pontryagin numbers: $p_1(TY) = c_1^2(TY) - 2c_2(TY)$

$$p_2(TY) = c_2^2(TY) - 2c_1(TY)\,c_3(TY) + 2c_4(TY) .$$

Fivefolds? Not generally true that Chern #s are inv. under crepant birational maps.

$$\underbrace{\int c_1^5,\ \int c_1^3 c_2,\ \int c_1^2 c_3,\ \int c_1 c_4,\ \int c_5}_{\text{inv.}},\ \int c_1 c_2^2,\ \int c_2 c_3$$

$\boxed{n_H^{ch}}$ 5d prepotentials = triple intersection polynomials.

The scalar fields in the vector mult. are restricted to the Cartan subalgebra of the Lie group.

The charge of hyper is simply given by a weight of R.

Intriligator-Morrison-Seiberg '97.

$$6\mathcal{F}_{\text{pre}}(\phi) = \frac{1}{2}\left(\sum_{e} |\langle \alpha_e, \phi\rangle|^3 - \sum_i \sum_{\bar{w}\in R_i} n_{R_i} |\langle \bar{w}, \phi\rangle|^3\right)$$

for all simple group (w/ the exception of $SU(N\geq 3)$), this is the full prepotential.

D_a : fibral divisors of an φ over B

$\Rightarrow$ irred. comp. of $\quad \varphi^* \underline{S} = \sum_a \underline{m_a} \, D_a$

D_0 = the divisor touching the section of the φ.

define a poly. over the Chow ring of Y $\qquad \varphi_* \left(\sum_a D_a \phi_a\right)^3$ in $A_*(B)[\phi_a]$

$$\Rightarrow \int_Y \left(\sum_a D_a \phi_a\right)^3 \cdot \phi^3 \mathcal{M} = \int_B \varphi_* \left(\sum_a D_a \phi_a\right)^3 \cdot \mathcal{M}.$$

Triple intersection poly : $6\mathcal{F}_{trip}(K, S, \phi) = \underbrace{\varphi_* \left(\left(\sum_a D_a \phi_a\right)^3\right.}_{} \cdot \left.\cancel{\varphi^*}\mathcal{M}_1\right)$

ex) G_2-model. $\qquad A_{\dim B - 2}(B)$

$$6\mathcal{F}_{pre} = -8\phi_1 (n_{14} + n_7 - 1) + 9\phi_2\phi_1^2 (-2n_{14} + n_7 + 2) + 3\phi_2^2\phi_1 (8n_{14} - n_7 - 8)$$
$$- 8(n_{14} - 1)\phi_2^3$$

$2 - 2g = -K \cdot S - S^2$ (g = genus of S)

$\| \; (\phi_0 \to 0)$

$$\begin{cases} n_7 = -10(g-1) + 3S^2 \\ n_{14} = g \end{cases}$$

$$6\tilde{\mathcal{F}}_{trip} = \underline{-8(g-1)\phi_0^3 + 3\phi_0^2\phi_1(4g - 4 - S^2) - 3\phi_0\phi_1^2(2g - 2 - S^2)}$$
$$- 8(g-1)\phi_1^3 + 24(3g - 3 - S^2)\phi_2^3 - 27(4g - 4 - S^2)\phi_1^2\phi_2 + 9(g - 6 - S^2)\phi_1\phi_2^2$$

Algorithm to get Geometric Data

Step 1: Determine a singular Weierstrass model w/ Kodaira fibers associated to the desired G.

Step 2: Determine a crepant resolution of the singular Weierstrass model.
(as a sequence of blowups.)

Step 3: Compute the pushforward formulae to push the total Chern class of the resolved elliptic fibration to its base.

→ generating fcn of χ is computed

→ for d-dim base, χ is given by the coeff. of t^d in power series expansion.

→ $\chi(CY_3)$

Step 4: Compute the Hodge #s

Step 5: Determine the fiber structure of the resolved Weierstrass model.

Step 6: Determine the representations by computing the geometric weights of the irred. comp. of the singular fibers over codim 2 pts.

Step 7: Compute the triple intersection poly.

5d $\mathcal{N}=1$ theory

$$n_V = h^{1,1}(Y)-1$$

$$n_H = n_H^0 + n_H^{ch} = h^{2,1}(Y)+1 + \sum_i n_{R_i}(\dim R_i - \dim R_i^{(0)})$$

$$\boxed{\begin{array}{l} \text{6d } \mathcal{N}=(1,0) \text{ theory} \\ \quad n_V = \dim G, \qquad n_T = 9-k^2 \\ \quad n_H = \text{the same}. \end{array}}$$

Anomaly Cancellations in 6d $\mathcal{N}=(1,0)$ supergravity.

Green-Schwarz mechanism for 6d. $\Rightarrow$ I_8. R, F_i

$\to$ pure-gravitational anomaly ($\mathrm{tr}R^4$ term)

$$n_H - n_V + 29 n_T - 273 = 0$$

The remainder of I_8:

$$I_8 = \frac{K^2}{8}(\mathrm{tr}R^2)^2 + \frac{1}{6}\sum_a X_a^{(2)}\,\mathrm{tr}R^2 - \frac{2}{3}\sum_a X_a^{(4)} + 4\sum_{a<b} Y_{ab}$$

$$X_a^{(n)} = \mathrm{tr}_{adj} F_a^{\,n} - \sum_i n_{R_i,a}\,\mathrm{tr}_{R_i,a} F_a^{\,n}.$$

$$Y_{ab} = \sum \underbrace{n_{R_i,a\ R_j,b}}_{(R_{i,a},\,R_{j,b})\ \ G_a\times G_b.}\ \mathrm{tr}_{R_i,a} F_a^2\ \mathrm{tr}_{R_j,b} F_b^2.$$

$$= \sum_{i,j} n_{R_i,a\,R_j,b}\, A_{R_i,a}\, A_{R_j,b}\ \mathrm{tr}_{F_a} F_a^2\ \mathrm{tr}_{F_b} F_b^2.$$

Consider $\underline{(R_1, R_2)}$ $\quad G = G_1 \times G_2$.

$$\dim R_2 \Rightarrow n_{R_1}$$
$$\dim R_1 \Rightarrow n_{R_2}$$

$$: \begin{cases} n_{R_1} = \cdots + \dim R_2 \, n_{R_1 R_2} \\ n_{R_2} = \cdots + \dim R_1 \, n_{R_1 R_2} \end{cases}$$

$\underline{\text{Hypers in } R_i}$: $\underbrace{\dim R_i - \dim R_i^{(0)}}$.

$$n_H^{ch} = \sum_i n_{R_i} (\dim R_i - \dim R_i^{(0)})$$

$$\mathrm{tr}_{R_i,a} F_a^2 = A_{R_i,a} \, \mathrm{tr}_{F_a} F_a^2$$

$$\mathrm{tr}_{R_i,a} F_a^4 = B_{R_i,a} \, \mathrm{tr}_{F_a} F_a^4 + C_{R_i,a} (\mathrm{tr}_{F_a} F_a^2)^2$$

$$X_a^{(2)} = (A_{a,adj} - \sum_i n_{R_i a} A_{R_i,a}) \, \mathrm{tr}_{F_a} F_a^2$$

$$X_a^{(4)} = \underbrace{(B_{a,adj} - \sum_i n_{R_i a} B_{R_i,a}) \, \mathrm{tr}_{F_a} F_a^4}_{=0} + (C_{a,adj} - \sum_i n_{R_i a} C_{R_i,a}) (\mathrm{tr}_{F_a} F_a^2)^2$$

pure gauge anomalies to cancel.

$G = G_1 \times G_2$

rearrange:

$$I_8 = \frac{K^2}{8}(\mathrm{tr}R^2)^2 + \frac{1}{6}\left(X_1^{(2)} + X_2^{(2)}\right)\mathrm{tr}R^2 + \frac{2}{3}\left(X_1^{(4)} + X_2^{(4)}\right) + 4Y_{12}$$

factors as $\frac{1}{2}\Omega_{ij} X_i^{(4)} X_j^{(4)}$, then the anomalies are cancelled by adding the counter term $\Omega_{ij} B_i \wedge X_j^{(4)}$ to the Lagrangian.

$$H^{(i)} = dB^{(i)} + \omega^{(i)}$$

g	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
λ	1	2	1	2	6	12	60	6	2

$$H = dB + \frac{1}{2}K\underbrace{\omega_{3L}}_{\text{SYM}} + 2\sum_\alpha \frac{S_\alpha}{\lambda_\alpha}\underbrace{\omega_{\alpha,3Y}}_{\text{CS}}$$

$I_8 = X\cdot X$. by adding $\Delta L_{GS} \propto \frac{1}{2}B\wedge X$.

$$X = \frac{1}{2}K\,\mathrm{tr}R^2 + \sum_\alpha \frac{2}{\lambda_\alpha} S_\alpha\, \mathrm{tr}F_\alpha^2.$$

the Dynkin index of F_α of G_α.

Coalesce :

$$n_T = 9 - K^2$$

$$n_H - n_V + 29 n_T - 273 = 0$$

$$B_{a,adj} - \sum_i n_{R_i,a} B_{R_i,a} = 0$$

$$\lambda_a \left(A_{a,adj} - \sum_i n_{R_i,a} A_{R_i,a} \right) = 6K \cdot \underline{S_a}$$

$$\lambda_a^2 \left(C_{a,adj} - \sum_i n_{R_i,a} C_{R_i,a} \right) = -3 S_a^2$$

$$\lambda_a \lambda_b \sum_{i,j} n_{R_i,a\,R_j,b} A_{R_i,a} A_{R_j,b} = S_a \cdot S_b \quad (a \neq b)$$

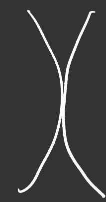
$SU(2) \times G_2$- model

$I_2^{s} + I_0^{*ns}$

$I_2^{ns} + I_0^{*ns}$

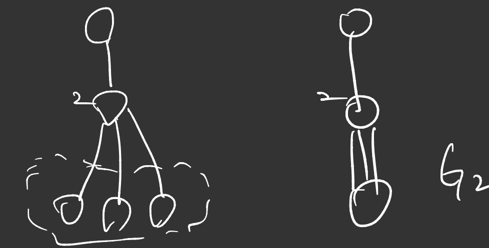
$III + I_0^{*ns}$

$IV^{ns} + I_0^{*ns}$



III

$S = V(s)$



I_0^{*ns}

$T = V(t)$.

$$y^2 z = x^3 + f\, \underline{s t^2}\, x z^2 + g\, \underline{s^2 t^3}\, z^3 .$$

$$\Delta = s^3 t^6 (4f^3 + 27 g^2 s)$$

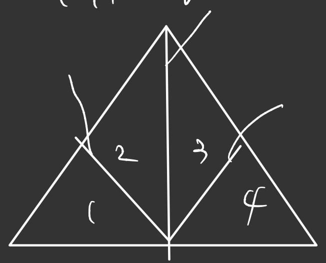
4 crepant resolutions.

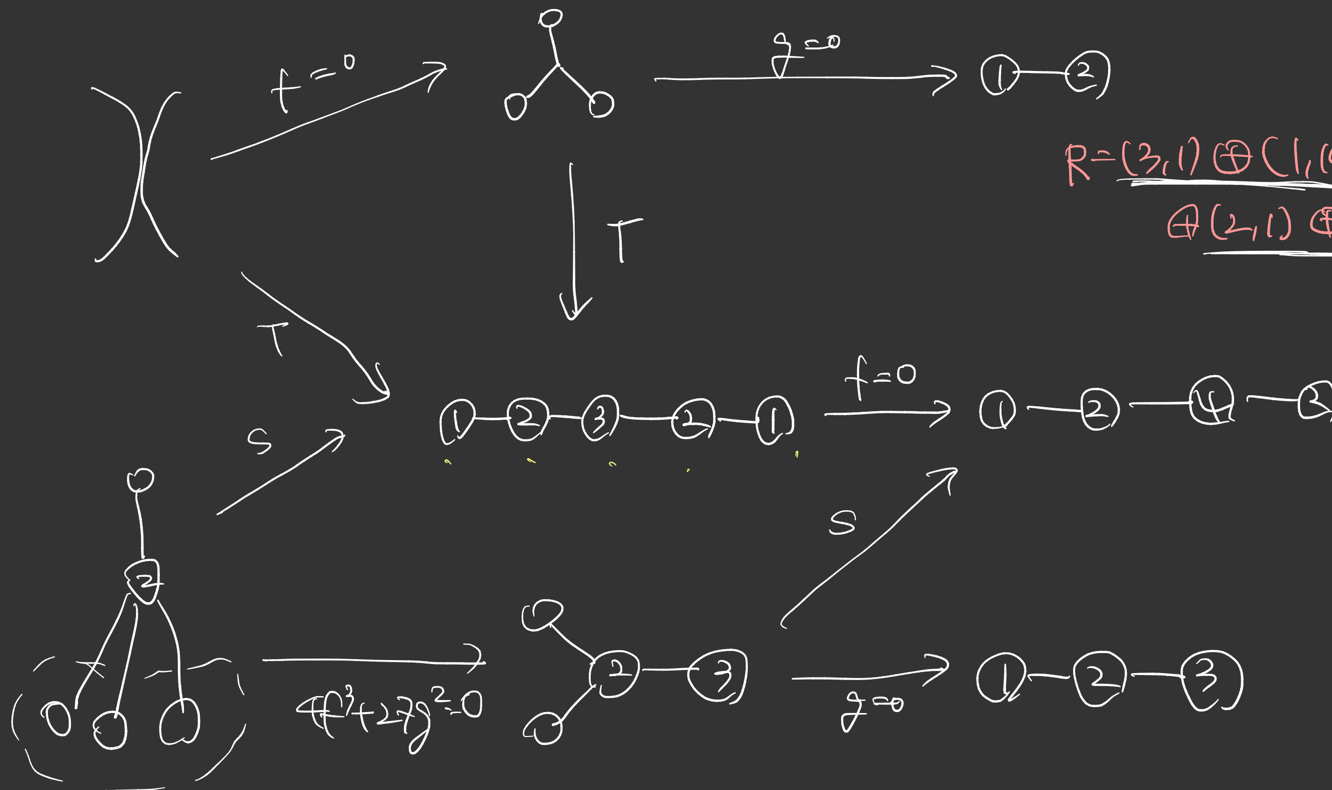
$$X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(x,y,t|w_1)} X_2 \xleftarrow{(y,w_1|w_2)} X_3$$

$$\chi(Y_3) = -6\,(10K^2 + S^2 + 5SK + 2ST + 8KT + 2T^2)$$

	$h^{1,1}$	$h^{2,1}$
$SU(2)$	$12 - K^2$	$12 + 29K^2 + 15KS + 3S^2$
G_2	$13 - K^2$	$13 + 29K^2 + 24KT + 6T^2$
$SU(2) \times G_2$	$14 - K^2$	$14 + 29K^2 + 15KS + 3S^2 + 24KT + 6T^2 + 6ST$.

(2,7) weights





$R = (3,1) \oplus (1,(4)) \oplus (2,7)$
$\oplus (2,1) \oplus (1,7)$

codim 1

2 $\quad$ $\mathrm{III}^* \cong E_7$

3 $\quad$ $\mathrm{IV}^* \cong E_6$

5d

$$n_{2,7} = \frac{1}{2} ST.$$

$$n_{1,7} = -T(K+S+2T)$$

$$n_{1,14} = \frac{1}{2}(KT + T^2 + 2)$$

$$\underline{n_{2,1} + 8\, n_{3,1}} = -S(4K - 2S - \tfrac{7}{2}T).$$

$$\Delta = s^3 t^6 (4f^3 + 27 g^2 s).$$

(2,1) $\quad s$ and $\Delta' = 4f^3 + 27g^2 s.$

$$n_{2,1} \overset{?}{=} S\cdot[f] = -S(4K + S + 2T)$$

2 curves of III has a proj. line & a conic $\quad y^2 - t^2 s(fx + gst) = 0$

$[y:s:x]$

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2\, t^2 f \\ 0 & 1/2\, t^2 f & -t^2 g \end{pmatrix} = \frac{1}{4} t^4 f^2.$$

$$n_{2,1} \overset{?}{=} 2S\cdot[V(f)] = \underline{2S\cdot(4K + S + 2T)}$$

6d

$$n_H - n_V - 29\, n_T - 273 = 0$$

$$I_8 = \frac{1}{2}\Big(\frac{1}{2}K\, \mathrm{tr}R^2 + 2S\, \mathrm{tr}_2 F_2^2 + T\, \mathrm{tr}_7 F_2^2\Big)^2$$

$$n_{2,1} = -S(8K + 2S + \tfrac{7}{2}T).$$

$$n_{3,1} = \frac{1}{2}(KS + S^2 + 2)$$

$n_{2,7}$, $n_{1,7}$, $n_{1,14}$ the same.

half-hypers in (2,7) affecting Δ contributes $1/2\, ST$.

$n_{2,1} = -2S\cdot(4K + S + 2T) + 1/2\, ST.$ ✓