Hybrid Statistics of a Random Model of Zeta over Intervals of Varying Length

Christine K. Chang
CUNY, The Graduate Center

CRG L-functions seminar
Oct 8th, 2024

What is the Riemann zeta function?

The Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

where $s = \sigma + it \in \mathbb{C}$, p is prime and $\Re(s) > 1$.

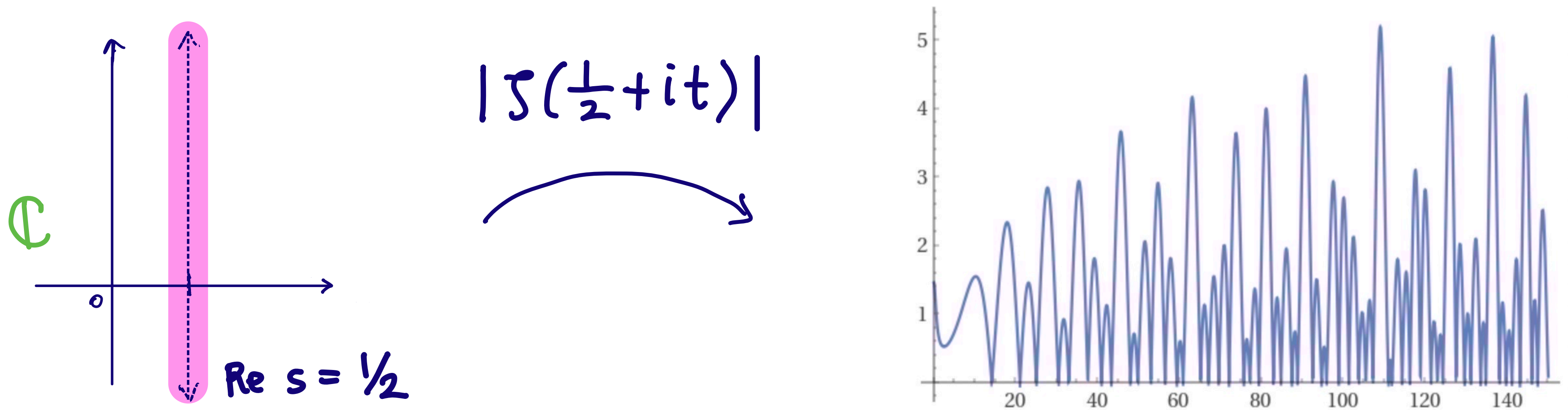
The Riemann Zeta Function can be analytically extended to $\mathbb{C} \setminus \{1\}$ by the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

trivial zeroes are at $s = -2n$ where $n = 1, 2, 3...$
Riemann's Hypothesis: all non-trivial zeroes lie on the $1/2$ line.

Main interest: extreme values of zeta

We are interested in the extreme values of the Riemann Zeta Function.

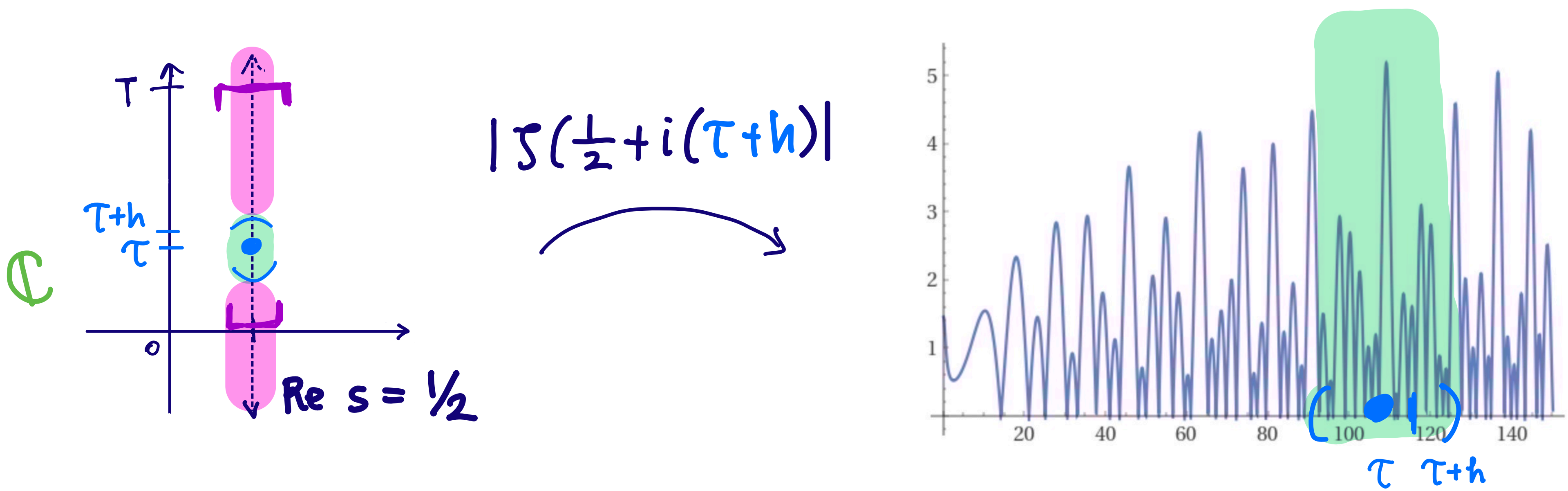


Lindelöf's Hypothesis: $\zeta(1/2 + it) = O(t^{\epsilon})$ for every positive ϵ. We do not know the correct order of magnitude of the global maximum – this is a hard problem!

Simpler question: study the local maximum over a short interval.

Choose $\tau \sim U[0, T]$, where T is a large value on the $1/2$ line. Consider a neighborhood $\mathcal{I}$ of varying length around τ. Let $h \in \mathcal{I}$.

We want to study $\log |\zeta(1/2 + i(\tau + h))|$ over short intervals.



We can study the local maximum over different interval sizes...

$$\max_{|h| \leq \frac{1}{\log T}} |\zeta(1/2 + i(\tau + h))| = ??$$

$$\max_{|h| \leq 1} |\zeta(1/2 + i(\tau + h))| = ??$$

$$\max_{|h| \leq (\log T)^\theta} |\zeta(1/2 + i(\tau + h))| = ??$$

The Fyodorov-Hiary-Keating Conjecture

Fyodorov-Hiary-Keating (2012) Conjecture:

$$\max_{|h|\leq 1} |\zeta(1/2 + i(\tau + h))| = \frac{\log T}{(\log\log T)^{3/4}} e^{\mathcal{M}_T(\tau)}$$

where $\mathcal{M}_T(\tau) \to \mathcal{M}$ as $T \to \infty$ and

$$\mathbb{P}(\mathcal{M} > y) \ll y e^{-2y}$$

as $y \to \infty$.

Connection to probability:
maximum of branching random walks
(collection of log-correlated random variables): same leading and subleading order!

The Fyodorov-Hiary-Keating Conjecture

Fyodorov-Hiary-Keating (2012) Conjecture:

$$\max_{|h|\leq 1} |\zeta(1/2 + i(\tau + h))| = \frac{\log T}{(\log \log T)^{3/4}} e^{\mathcal{M}_T(\tau)}$$

where $\mathcal{M}_T(\tau) \to \mathcal{M}$ as $T \to \infty$ and

$$\mathbb{P}(\mathcal{M} > y) \ll y e^{-2y}$$

as $y \to \infty$.

Connection to probability:

maximum of branching random walks

(collection of log-correlated random variables): same leading and subleading order!

maximum of IID random variables

(a collection of independent and identically distributed random variables (IID)): same leading order, but different subleading order exponent of $1/4$.

Proving the Fyodorov-Hiary-Keating Conjecture

Arguin-Bourgade-Radziwill (2020, 2023):

$$\mathbb{P}\left(\max_{|h|\leq 1}|\zeta(1/2+i(\tau+h)|>\frac{\log T}{(\log\log T)^{3/4}}e^y\right)\asymp y e^{-2y}e^{-y^2/\log\log T}$$

Arguin-Bailey (2022): Let $\theta>0$. Then

$$\mathbb{P}\left(\max_{|h|\leq(\log T)^\theta}|\zeta(1/2+i(\tau+h))|>\frac{(\log T)^{\sqrt{1+\theta}}}{(\log\log T)^{\frac{1}{4\sqrt{1+\theta}}}}e^y\right)\ll 1e^{-2\sqrt{1+\theta}y}e^{-y^2/\log\log T}$$

When $\theta\to 0$, the right tail distribution and subleading orders are different! Why? Recently, Arguin-Dubach-Hartung studied a random model of zeta and addressed the subleading order exponent discrepancy.

What is the random model of zeta?

$p^{-i\tau} = e^{-i\tau \log p}$ } IID and uniformly distributed on unit circle

Let $\tau \sim U[0, T]$ and $h \in \mathcal{I}$. Then

$$\log \zeta(1/2 + i(\tau + h)) \approx \log \prod_p \left(1 - \frac{1}{p^{1/2+i(\tau+h)}}\right)^{-1} \approx \sum_{p \le T} \frac{p^{-i\tau} p^{-ih}}{p^{1/2}}.$$

Identify $p^{-i\tau}$ with IID Gaussian processes $G_p \sim \mathcal{N}_{\mathbb{C}}(0, 1)$.

Since $\log |\zeta| = \Re \log \zeta$, we have $X_T(h) = \sum_{p \le T} \frac{\Re(G_p p^{-ih})}{\sqrt{p}}$.

What is the random model of zeta?

Let $\tau \sim U[0, T]$ and $h \in \mathcal{I}$. Then

$$\log \zeta(1/2 + i(\tau + h)) \approx \log \prod_p \left(1 - \frac{1}{p^{1/2+i(\tau+h)}}\right)^{-1} \approx \sum_{p \le T} \frac{p^{-i\tau} p^{-ih}}{p^{1/2}}.$$

Identify $p^{-i\tau}$ with Gaussian processes $G_p \sim \mathcal{N}_{\mathbb{C}}(0, 1)$.

Since $\log|\zeta| = \Re \log \zeta$, we have $X_T(h) = \sum_{p \le T} \frac{\Re(G_p p^{-ih})}{\sqrt{p}}$.

The random variable $X_T(h) \sim \mathcal{N}\left(0, \frac{\log\log T}{2} + O(1)\right)$.

Mertens' Estimate

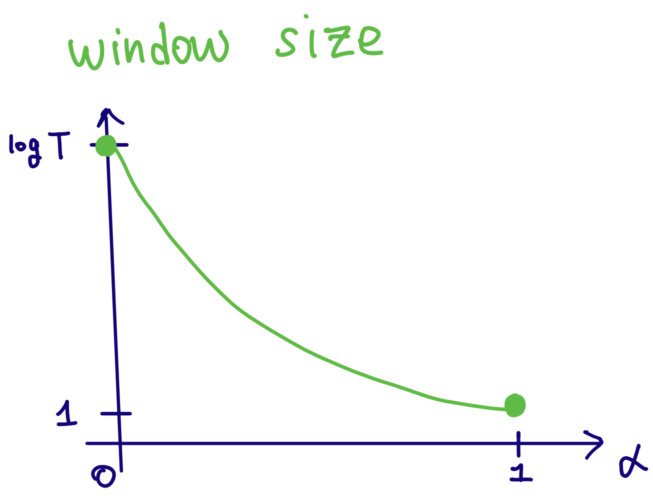
$$\mathbb{E}\left[(X_T(h))^2\right] = \sum_{p \le T} \frac{\mathbb{E}\left[(\Re(G_p))^2\right]}{p} = \frac{1}{2} \sum_{p \le T} \frac{1}{p} = \frac{\log\log T}{2} + O(1)$$

Intervals of Varying Length

Define $\theta = (\log \log T)^{-\alpha}$ where $\alpha \in (0, 1)$. We study the process $(X_T(h), h \in \mathcal{I})$ over intervals $\mathcal{I} = [-(\log T)^\theta, (\log T)^\theta]$. This implies

$$|\mathcal{I}| = 2(\log T)^\theta = 2\exp((\log \log T)^{1-\alpha}).$$

As α ranges between 0 and 1, the size of the window ranges between 1 and $\log T$.



Covariance structure of the random model

For all $h, h' \in \mathcal{I}$, we have the following covariance structure of the process:

$$\begin{aligned} &\mathbb{E}[X_T(h), X_T(h')] \\ &= \frac{1}{2}\sum_{p\le T}\frac{\cos(|h-h'|\log p)}{p} \\ &= \frac{1}{2}\sum_{p\le \exp(|h-h'|^{-1})}\frac{\cos(|h-h'|\log p)}{p} + \sum_{\exp(|h-h'|^{-1})<p<\exp(e^t)}\frac{\cos(|h-h'|\log p)}{p} \end{aligned}$$

that depends on the distance between the two points:

$$\mathbb{E}[X_T(h)X_T(h')] = \begin{cases} \frac{1}{2}\log|h-h'|^{-1} + O(1) & \text{if} \quad |h-h'| \le 1 \\ O(|h-h'|^{-1}) & \text{if} \quad |h-h'| > 1 \end{cases}$$

Intervals of size one: $X_T(h), X_T(h')$ are log correlated!

Larger intervals: $X_T(h), X_T(h')$ are weakly correlated!

We expect hybrid statistics as the interval size varies in length.

Random Model results

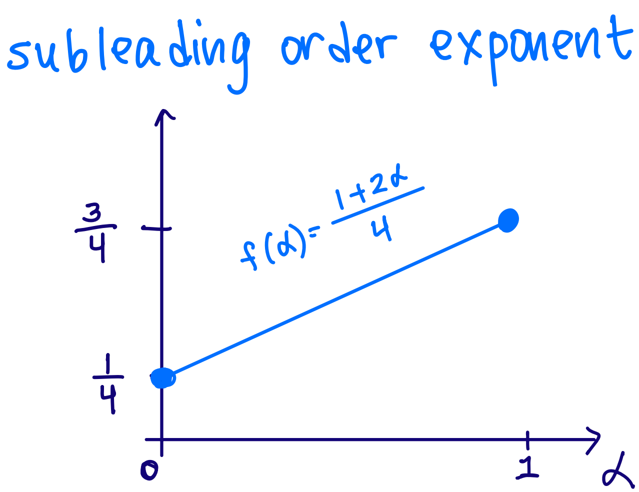
$X_T(h)$ denotes the **random model** of $\log|\zeta(1/2+i(\tau+h))|$.

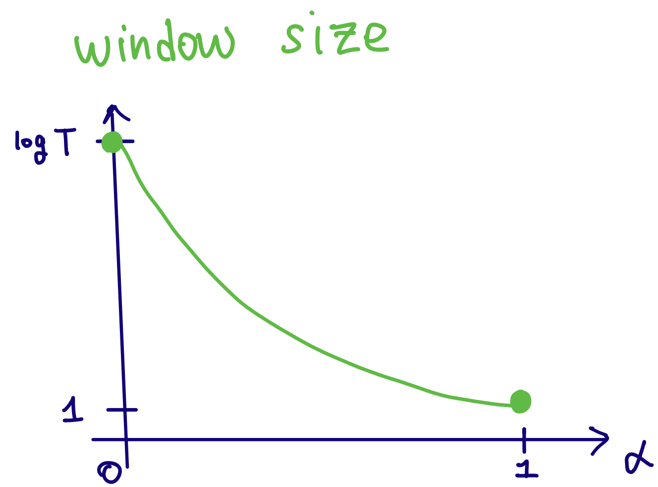
This implies $e^{X_T(h)} \approx |\zeta(1/2+i(\tau+h))|$.

Arguin-Dubach-Hartung (2024):

$$\mathbb{P}\left(\max_{|h|\le(\log T)^{\theta}} e^{X_T(h)} > \frac{(\log T)^{\sqrt{1+\theta}}}{(\log\log T)^{\frac{1+2\alpha}{4\sqrt{1+\theta}}}} e^{g(T)}\right) = o(1),$$

with $g(T)\to\infty$ arbitrarily slowly, and $\theta = (\log\log T)^{-\alpha}$ where $\alpha\in(0,1)$.





The main result

Theorem (C. 2024)

Let $y \in \mathbb{R}_+$ and $y = O(\frac{\log\log T}{\log\log\log T})$. Let $\theta = (\log\log T)^{-\alpha}$ with $\alpha \in (0,1)$. Then we have

$$\mathbb{P}\left(\max_{|h|\le(\log T)^\theta} e^{X_T(h)} > \frac{(\log T)^{\sqrt{1+\theta}}}{(\log\log T)^{\frac{1+2\alpha}{4\sqrt{1+\theta}}}} e^y\right) \asymp \left(1+\frac{y}{(\log\log T)^{1-\alpha}}\right) e^{-2\sqrt{1+\theta}y} e^{-\frac{y^2}{\log\log T}}.$$

$\alpha = 0$	$\alpha \in (0,1)$	$\alpha = 1$
IID	hybrid regime	log-correlated
subleading order: $\frac{1}{4}$	subleading order: $\frac{1+2\alpha}{4}$	subleading order: $\frac{3}{4}$
$\ll 1 e^{-2y} e^{-y^2/\log\log T}$	$\ll \left(1+\frac{y}{(\log\log T)^{1-\alpha}}\right) e^{-2\sqrt{1+\theta}y} e^{-y^2/\log\log T}$	$\ll y e^{-2y} e^{-y^2/\log\log T}$

Moment over intervals of varying length

By the main result, we prove the following corollary for the $\beta_c = 2\sqrt{1+\theta}$ moment of a random model of $|\zeta(\frac{1}{2} + i(\tau + h))$.

Corollary

For $\beta_c = 2\sqrt{1+\theta}$ and $\alpha \in (0,1)$, we have for $A > 0$,

$$\int_{|h| \le (\log T)^\theta} (\exp \beta_c X_T(h))\, dh \ll_A \frac{(\log T)^{\frac{\beta_c^2}{4}+\theta}}{(\log \log T)^{\alpha - 1/2}}, \tag{1}$$

for all $t \in [T, 2T]$ except possibly on a subset of Lebesgue measure $\ll 1/A$.

Moment over intervals of varying length

By the main result, we prove the following corollary for the $\beta_c = 2\sqrt{1+\theta}$ moment of a random model of $|\zeta(\frac{1}{2} + i(\tau + h))$.

Corollary

For $\beta_c = 2\sqrt{1+\theta}$ and $\alpha \in (0,1)$, we have for $A > 0$,

$$\int_{|h| \leq (\log T)^{\theta}} (\exp \beta_c X_T(h)) \, dh \ll_A \frac{(\log T)^{\frac{\beta_c^2}{4}+\theta}}{(\log\log T)^{\alpha - 1/2}}, \tag{1}$$

for all $t \in [T, 2T]$ except possibly on a subset of Lebesgue measure $\ll 1/A$.

Theorem (Harper 2019)

Uniformly for all large T, we have

$$\int_{|h| \leq 1} |\zeta(1/2 + it + ih)|^2 dh \leq A \frac{\log T}{\sqrt{\log\log T}}$$

for all $t \in [T, 2T]$ except possibly on a subset of Lebesgue measure $\ll \frac{(\log A) \wedge \sqrt{\log\log T}}{A}$

Moment over intervals of varying length

Following the techniques of Arguin-Bailey, a sharper bound for $\alpha \in (0, 1/2]$ holds:

$$\int_{|h| \le (\log T)^{\theta}} |\zeta(1/2 + i(\tau + h)|^{\beta_c} \, dh \ll A(\log T)^{\frac{\beta_c^2}{4} + \theta}. \tag{2}$$

This holds for all $t \in [T, 2T]$ except possibly on a subset of Lebesgue measure $\ll 1/A$.

What is special about the correction?

Let's compare:

$\alpha \in (0, 1/2)$	$\alpha \in (1/2, 1)$	$\alpha = 1$
regime: IID	regime: hybrid	regime: log-correlated
correction: 1	correction: $\frac{1}{(\log\log T)^{\alpha - 1/2}}$	correction: $\frac{1}{\sqrt{\log\log T}}$

There is a distinctive transition at $\alpha = 1/2$ to the IID regime, i.e. for intervals that have length greater than $(\exp(\sqrt{\log\log T})$.

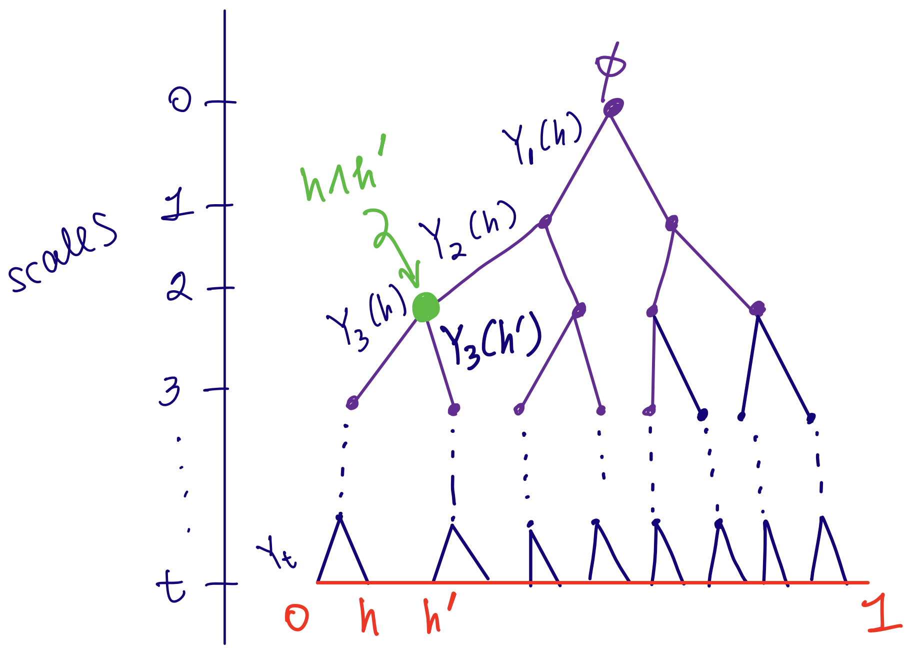
Proving the main result: first identify a BRW in the model

Recall that

$$\mathbb{E}[X_T(h)X_T(h')] = \begin{cases} \frac{1}{2}\log|h-h'|^{-1} + O(1) & \text{if } |h-h'| \le 1 \\ O(|h-h'|^{-1}) & \text{if } |h-h'| > 1 \end{cases}$$

Interval of order one: the random variables are log-correlated!

A branching random walk is a collection of log-correlated random variables!



Binary tree of depth t.

Increments $Y_\ell \sim$ IID $\mathcal{N}(0,\sigma^2)$

Gaussian Process $(X_t(h),\ h \in \mathcal{H}_t)$

$\mathcal{H}_t$ = leaves of a binary tree of depth t.

$X_t(h) = \sum_{\ell=0}^{t} Y_\ell(h) = Y_1(h) + Y_2(h) + \cdots + Y_t(h)$

Variance $\mathbb{E}[X_t^2(h)] = \sum_{\ell=0}^{t} \mathbb{E}[Y_\ell^2(h)] = \sigma^2 t$

Covariance $\mathbb{E}[X_t(h)X_t(h')] = \sigma^2 (h \wedge h')$

$h \wedge h'$ = branching time of $X_\ell(h)$ and $X_\ell(h')$

Approximate BRW in the random model of zeta

Recall that the random model of log $|\zeta|$ is

$$X_T(h) = \sum_{p \le T} \frac{\mathrm{Re}(G_p p^{-ih})}{\sqrt{p}}$$

NOT a sum of IID increments! Variance is different for each term.

Approximate BRW in the random model of zeta

Recall that the random model of $\log|\zeta|$ is

$$X_T(h) = \sum_{p \le T} \frac{\mathrm{Re}(G_p p^{-ih})}{\sqrt{p}}$$

NOT a sum of IID increments! Variance is different for each term.

Correct Increment

$$Y_K(h) = \sum_{e^{K-1} < \log p < e^K} \frac{\mathrm{Re}(G_p p^{-ih})}{\sqrt{p}}$$

$$\mathrm{Var}(Y_K(h)) = \mathbb{E}[Y_K^2(h)] = \frac{1}{2} \sum_{e^{K-1} < \log p < e^K} \frac{1}{p} \overset{\text{PNT}}{=} \frac{1}{2} \int_{e^{e^{K-1}}}^{e^{e^K}} \frac{1}{u \log u}\, du = \frac{1}{2} + o(1)$$

$$Y_K \sim \text{IID } \mathcal{N}\left(0, \tfrac{1}{2} + o(1)\right)$$

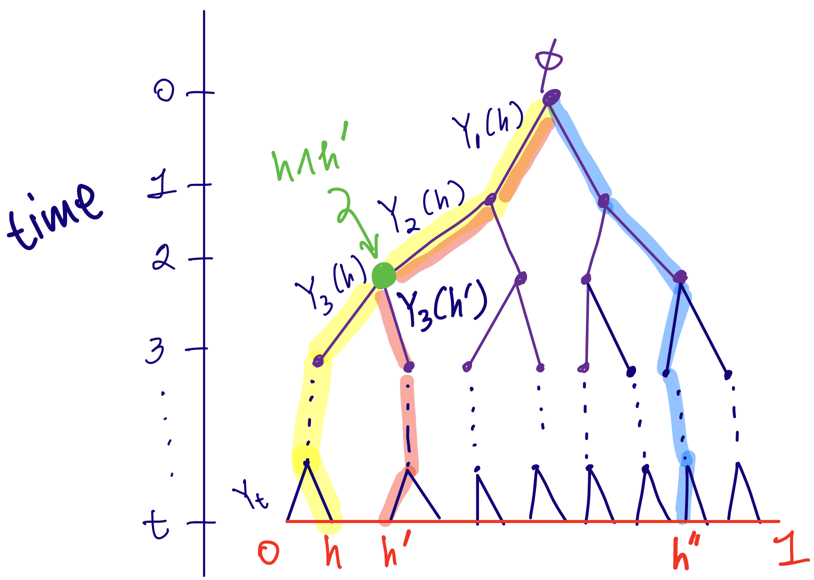
We can express the random model of zeta as a sum of IID increments (as a BRW).

$X_T(h)$ as an approximate BRW : $S_j(h) = \sum_{\ell=0}^{j} Y_\ell(h)$ } sum of IID increments

let $t = \log\log T$

$\mathrm{Var}(S_t(h)) = \mathbb{E}[S_t^2(h)] = \sum_{\ell=0}^{t} \mathbb{E}[Y_\ell(h)] = \frac{t}{2} + o(1)$.

Selberg's CLT: $\log|\zeta(\frac{1}{2}+it)| \sim \mathcal{N}(0, \frac{\log\log T}{2})$



Tree of degree $e \approx 2.718$ with $t = \log\log T$

Increments $Y_\ell \sim$ IID $\mathcal{N}(0, \frac{1}{2} + o(1))$

Gaussian Process $(X_t(h), h \in \mathcal{H}_t)$

Variance $\mathbb{E}[X_t^2(h)] = \sum_{\ell=0}^{t} \mathbb{E}[Y_\ell^2(h)] = \frac{\log\log T}{2} + o(1)$

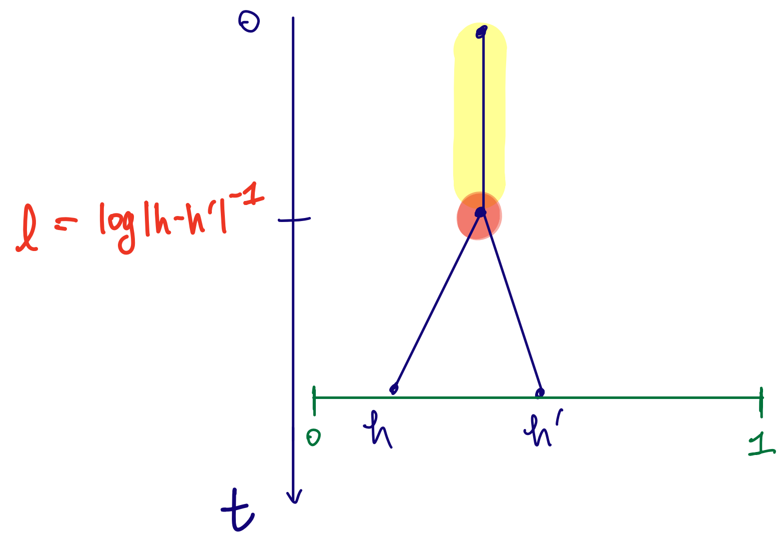
Covariance $\mathbb{E}[X_t(h) X_t(h')] = \sigma^2 (h \wedge h')$

$h \wedge h' =$ branching time of $X_\ell(h)$ and $X_\ell(h')$

Random model of zeta over an interval of order one

$h \wedge h' =$ branching time of $X_\ell(h)$ and $X_\ell(h')$

$h \wedge h' = \log|h-h'|^{-1}$.



covariance

$$E[X_t(h)X_t(h')] = \frac{1}{2}\log|h-h'|^{-1}$$

$$= \begin{cases} \frac{\ell}{2} + O(e^{2\ell}(|h-h'|^2) & \text{if } |h-h'| < e^{-\ell} \\ O(e^{-\ell}|h-h'|^{-1}) & \text{if } |h-h'| > e^{-\ell} \end{cases}$$

In short,

$S_\ell(h) \approx S_\ell(h')$ whenever $|h-h'| < e^{-\ell}$.

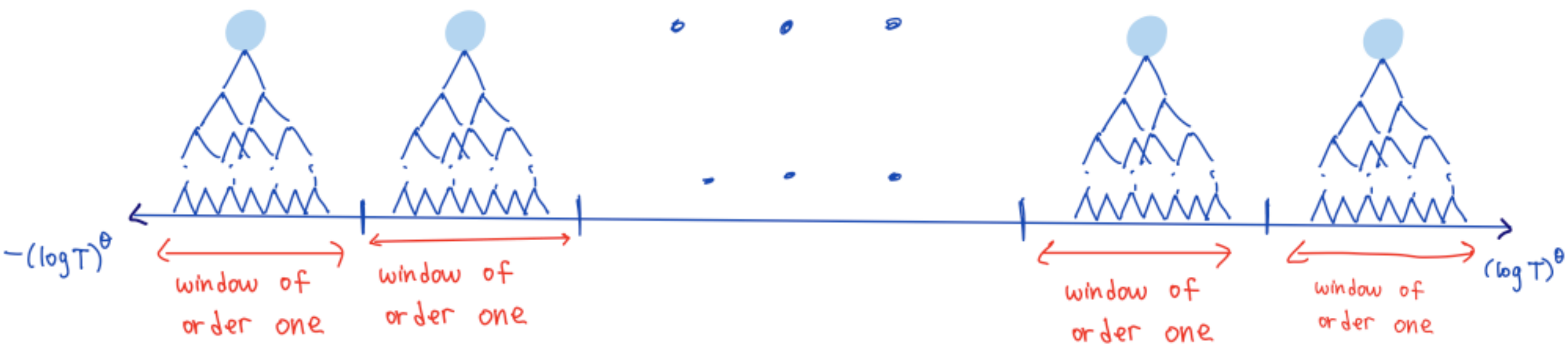
NT analogy: ζ values roughly change every $\frac{1}{\log T} = e^{-t}$

Big picture: analyze many independent BRW's

Consider once again the covariance:

$$\mathbb{E}[S_t(h)S_t(h')] = \begin{cases} \frac{1}{2}\log|h-h'|^{-1} + O(1) & \text{if } |h-h'| \le 1 \\ O(|h-h'|^{-1}) & \text{if } |h-h| > 1 \end{cases}$$

For $|h-h^{'}|>1$, there is a strong decoupling. We can think of the process $(S_t(h), |h| \le (\log T)^\theta)$ as behaving like $(\log T)^\theta$ independent copies of $(S_t(h), |h| \le 1)$.



We are essentially studying the extreme values of $(\log T)^\theta$ independent branching random walks!

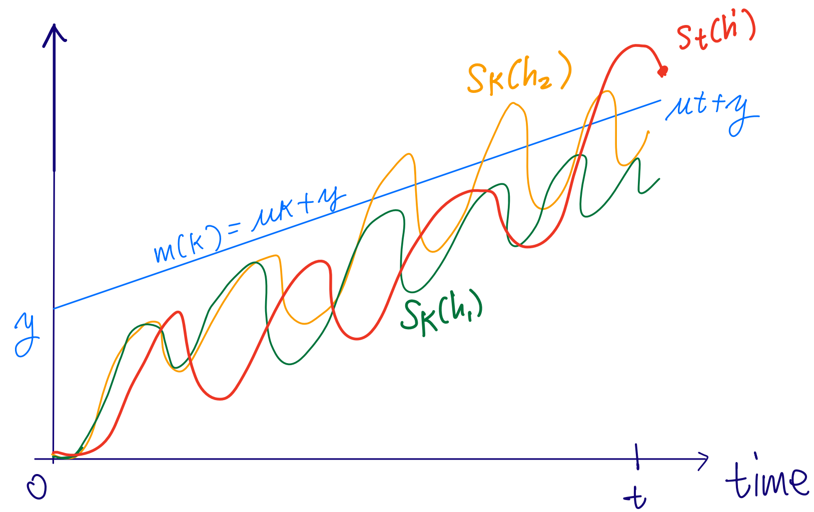
Proof ideas

We want an upper bound for

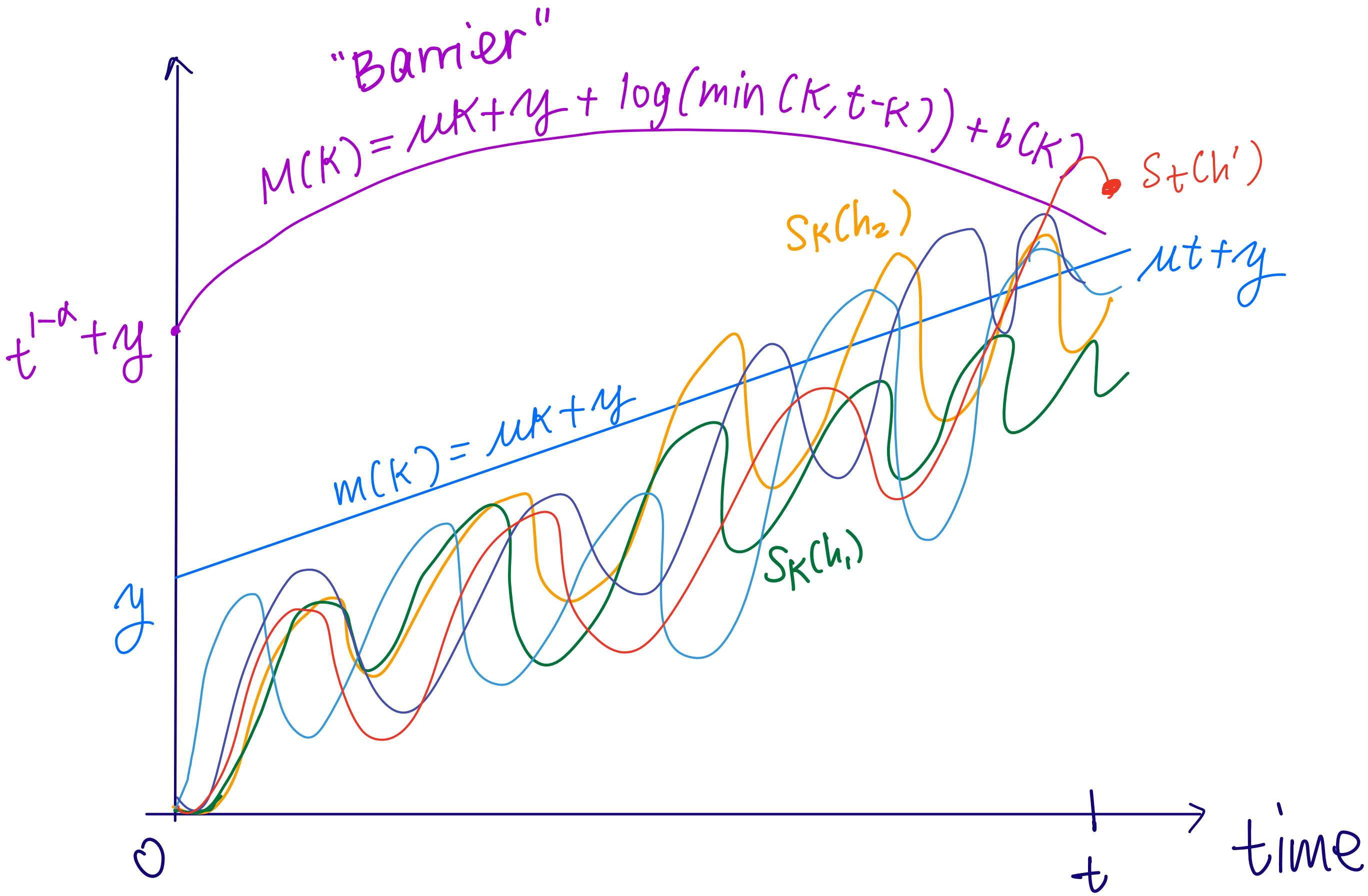
$$\mathbb{P}\left(\max_{|h|\le(\log T)^\theta} X_T(h) > \sqrt{1+\theta}\log\log T - \frac{1+2\alpha}{4\sqrt{1+\theta}}\log\log\log T + y\right).$$

Recall that $t = \log\log T$. Use the branching random walk model and consider

$$\mathbb{P}\left(\max_{|h|\le(\log T)^\theta} S_t(h) > \mu t + y\right), \quad \text{where } \mu = \sqrt{1+\theta} - \frac{1+2\alpha}{4\sqrt{1+\theta}}\frac{\log t}{t}.$$

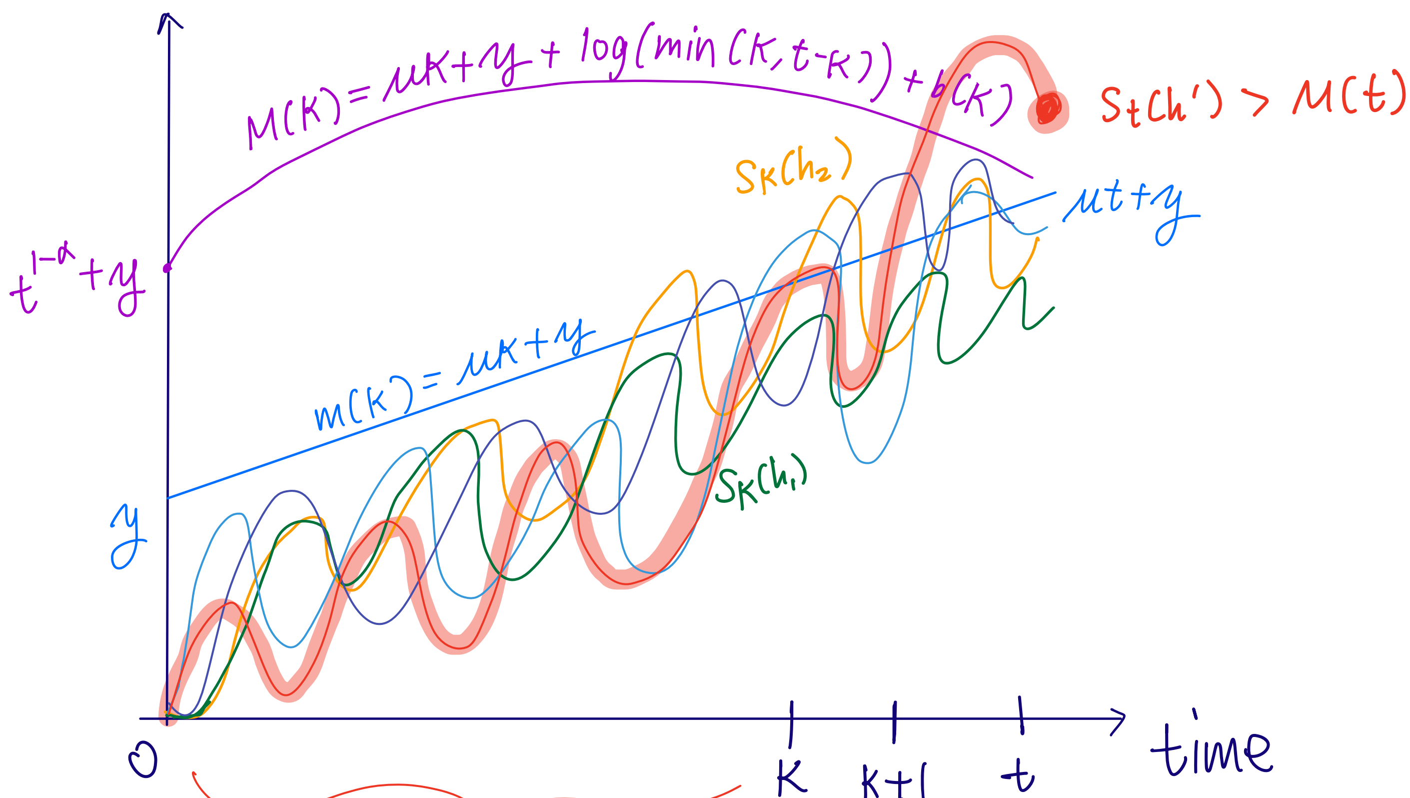


By self-similarity of BRW's, $\max_h S_K(h) \approx \mu K + y$. we can assume walks are below a "barrier".



The barrier also helps to remove rare events.

Construct a good event based on the "hitting time".



All walks are below the Barrier

$$\max_h S_j(h) < M(j), \forall j \leq k$$

S_{k+1} "hits" the barrier

S_t is above the barrier

Apply the hitting time method

Decompose the event over when the maximum of the process crosses the barrier, say at $k+1$:

$$\mathbb{P}\left(\max_{|h|\le(\log T)^\theta} S_t(h) > M(t)\right)$$
$$= \sum_{k=0}^{t-1} \mathbb{P}\Bigg(\max_{|h|\le(\log T)^\theta} S_t(h) > M(t), \quad \max_{|h|\le(\log T)^\theta} S_j(h) < M(j), \forall j \le k,$$
$$\max_{|h|\le(\log T)^\theta} S_{k+1}(h) > M(k+1).$$

We drop the first event and narrow our focus to the following sum:

$$\le \sum_{k=0}^{t-1} \mathbb{P}\left(\max_{|h|\le(\log T)^\theta} S_j(h) < M(j); \forall j \le k, \quad \max_{|h|\le(\log T)^\theta} S_{k+1}(h) > M(k+1)\right).$$

To obtain good estimates, we divide the range of k into two. Why?

Remove the barrier for a small range of k

First range of k (drop the barrier):

Discretize the interval and perform a union bound on $e^{k+t\theta}$ points.

By a Gaussian estimate, we have

$$\sum_{k=0}^{t-t^\alpha} \mathbb{P}\left(\max_{|h|\le(\log T)^\theta} S_{k+1}(h) > M(k+1)\right) \le \sum_{k=0}^{t-t^\alpha} e^{k+t\theta}\mathbb{P}\left(\max_{|h|\le e^{-k}} S_{k+1}(h) > M(k+1)\right)$$
$$\ll e^{-2\sqrt{1+\theta}y} e^{-y^2/t}$$

We discretize the interval of size $2(\log T)^\theta = 2e^{t\theta}$.

Divide by e^{-k} since $S_k(h) \approx S_k(h')$ whenever $|h-h'| < e^{-k}$.

Proof of main result: Estimate the second range of k

Second range (keep the barrier:) Discretize the interval as before and perform a union bound.

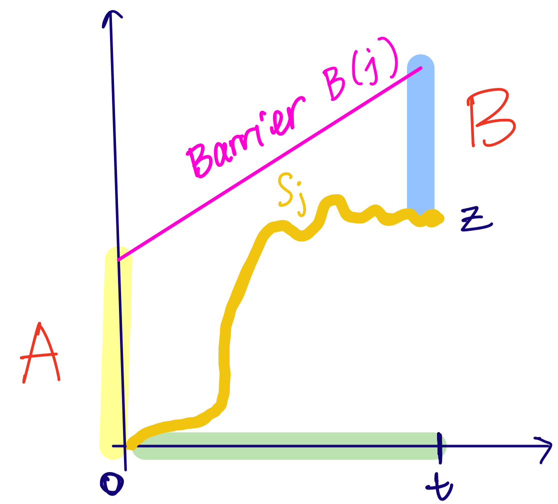
$$\sum_{k=t-t^{\alpha}}^{t} \mathbb{P}\left(\max_{|h|\leq(\log T)^{\theta}} S_j(h) < M(j); \forall j < k, \max_{|h|\leq(\log T)^{\theta}} S_{k+1}(h) > M(k+1) \right)$$

$$\leq \sum_{k=t-t^{\alpha}}^{t} e^{k+1+t\theta} \, \mathbb{P}\left(S_j(0) < M(j); \forall j < k, \;\; S_{k+1}(0) > M(k+1)\right)$$

Proof of main result: Estimate the second range of k

Ballot Theorem

Let $(S_j, j \le t)$ be a random walk with Gaussian increments of mean 0 and variance 1/2.
Define the barrier $B(j)$; $\forall j \le t$.
Let $z \le B(j)$ $\forall j \geq 1$.



$$P\left(S_j < \text{Barrier}; \forall j \le t, S_t \in (z-1, z] \right)$$

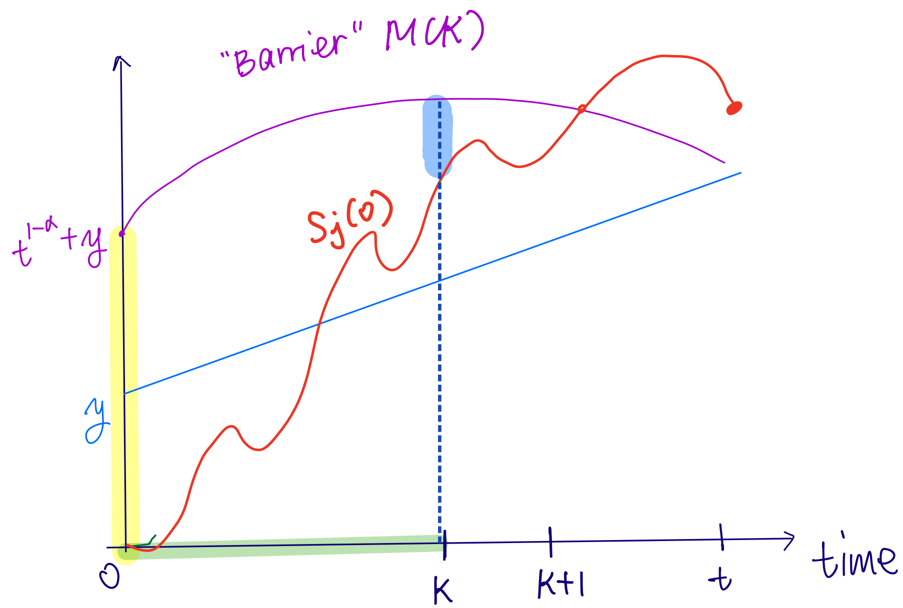
$$\ll \frac{A \cdot B}{\text{length of interval}} \cdot \frac{\exp\left(-\frac{z^2}{t}\right)}{t^{1/2}}$$

$$= \frac{AB}{t} \cdot \frac{\exp\left(-\frac{z^2}{t}\right)}{t^{1/2}}$$

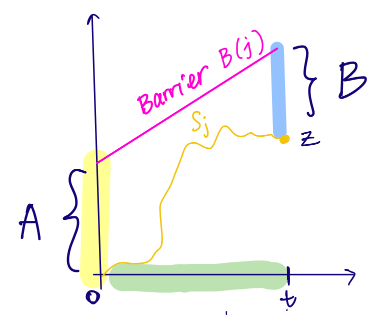
$$= \frac{AB}{t^{3/2}} \cdot \exp\left(-\frac{z^2}{t}\right)$$

Proof of main result: Estimate the second range of k

$P\left(S_j(0) < M(j);\ \forall j \le K,\ S_{K+1}(0) > M(K+1)\right) \le\ ?$



We want to use the Ballot Theorem!



Next Step: Decompose the event over all possible values of $S_K(0)$. Let's say $S_K(0) \in [u, u+1]$.

Decompose the event over all possible values u of the process at time k. The probability becomes

$$\mathbb{P}\left(S_j(0) < M(j); \forall j < k, \quad S_{k+1}(0) > M(k+1)\right)$$
$$= \sum_{u \leq M(k)} \mathbb{P}(S_j(0) < M(j); \forall j < k, S_k(0) \in [u, u+1])$$
$$\cdot \mathbb{P}(S_{k+1}(0) - S_k(0) > M(k+1) - u - 1)$$

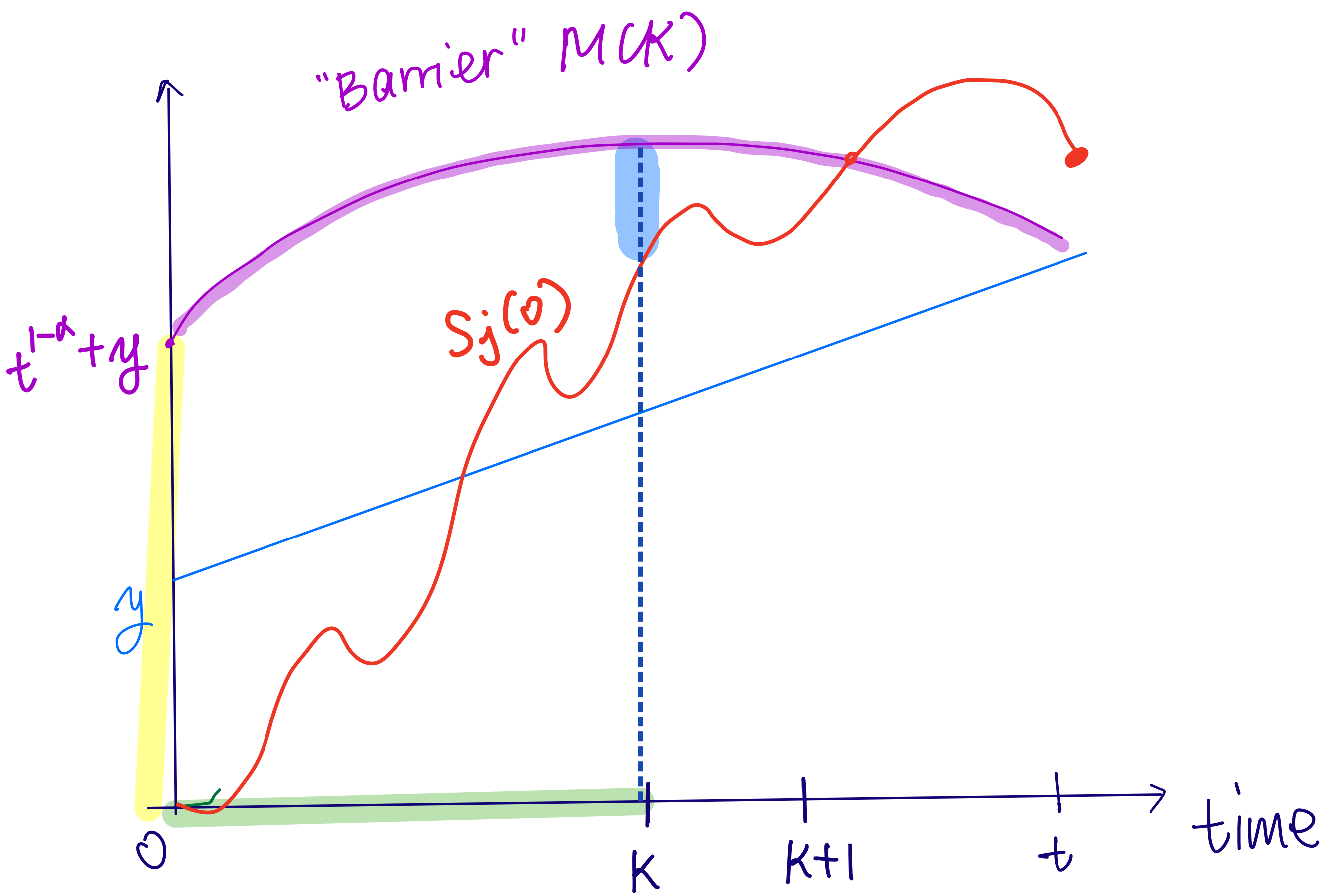
by independence of Gaussian increments.

By a Gaussian estimate, we have

$$\mathbb{P}(S_{k+1}(0) - S_k(0) > M(k+1) - u - 1) \ll e^{-(M(k+1)-u-1)^2}$$

By the Ballot Theorem, we have

$$\mathbb{P}(S_j(0) < M(j); \forall j < k, S_k(0) \in [u, u+1]) \ll \frac{(t^{1-\alpha} + y)(M(k) - u - 1)}{k} \cdot \frac{e^{-\frac{u^2}{k}}}{k^{1/2}}$$



Putting everything together and skipping some details, we have an expression

$$\sum_{k=k*}^{t} \mathbb{P}\left(\max_{|h|\le(\log T)^\theta} S_j(h) < M(j); \forall j < k, \max_{|h|\le(\log T)^\theta} S_{k+1}(h) > M(k+1) \right) \tag{3}$$

$$\le \sum_{k=k*}^{t} e^{k+1+t\theta}\, \mathbb{P}\left(S_j(0) < M(j); \forall j < k,\ \ S_{k+1}(0) > M(k+1)\right) \tag{4}$$

$$\ll \sum_{k=k*}^{t} e^{k+1+t\theta} \sum_{u\le M(k)} e^{-(M(k+1)-u-1)^2} \frac{e^{-\frac{u^2}{k}}}{k^{3/2}} (t^{1-\alpha}+y)(M(k)-u-1) \tag{5}$$

$$\vdots \tag{6}$$

$$\ll (t^{1-\alpha}+y)\exp(-2\mu y)\exp(-\frac{y^2}{t}) \sum_{k=t-t^\alpha}^{t} e^{-(t-k)\theta + \frac{1+2\alpha}{2}\log k - 2\beta\log(t-k)} k^{-3/2} \tag{7}$$

$$\ll t^{1-\alpha}(1+y/t\theta)\exp(-2\mu y)\exp(-y^2/t) \sum_{k=t-t^\alpha}^{t} k^{\alpha-1} e^{-(t-k)\theta - 2\beta\log(t-k)} \tag{8}$$

$$K \gg t \Rightarrow \frac{1}{K^{1-\alpha}} \ll \frac{1}{t^{1-\alpha}}$$

$$\vdots \tag{9}$$

$$\ll \left(1+\frac{y}{t^{1-\alpha}}\right)\exp(-2\sqrt{1+\theta}\,y)\exp\left(-\frac{y^2}{t}\right). \tag{10}$$

What does this mean for the Riemann zeta function??

Conjecture for the Riemann zeta function

Let $0 < \alpha < 1$ and $\theta = (\log\log T)^{-\alpha}$. Then

$$\max_{|h| \leq (\log T)^\theta} |\zeta(1/2 + i(\tau + h))| = \frac{(\log T)^{\sqrt{1+\theta}}}{(\log\log T)^{\frac{1+2\alpha}{4\sqrt{1+\theta}}}} e^{\mathcal{M}_T},$$

where τ is uniformly distributed on $[T, 2T]$, and $(\mathcal{M}_T, T > 1)$ is a tight sequence of random variables converging as $T \to \infty$ to a random variable $\mathcal{M}$ with right tail

$$(\mathcal{M} > y) \sim \left(1 + \frac{y}{(\log\log T)^{1-\alpha}}\right) e^{-2\sqrt{1+\theta} y} e^{-\frac{y^2}{\log\log T}}.$$

Conjecture for moments of zeta over short varying intervals

Let $\beta_c = 2\sqrt{1+\theta}$ and $\alpha \in (1/2, 1)$. Then we have for $A > 0$,

$$\left(\frac{1}{(\log T)^\theta} \int_{|h| \leq (\log T)^\theta} |\zeta(1/2 + i(t+h))|^{\beta_c} dh > A \frac{(\log T)^{\frac{\beta_c^2}{4}}}{(\log\log T)^{\alpha - 1/2}}\right) \ll \frac{1}{A}.$$

The End

Thank you for listening!

References

[1] Louis-Pierre Arguin. "Extrema of Log-correlated Random Variables: Principles and Examples". In: Cambridge University Press, 2017, pp. 166–204.

[2] Louis-Pierre Arguin and Emma Bailey. "Large Deviation Estimates of Selberg's Central Limit Theorem and Applications". In: *International Mathematics Research Notices* 2023.23 (2023), pp. 20574–20612.

[3] Louis-Pierre Arguin, Paul Bourgade, and Maksym Radziwiłł. "The Fyodorov-Hiary-Keating Conjecture. I". In: *arXiv preprint arXiv:2007.00988* (2020).

[4] Louis-Pierre Arguin, Paul Bourgade, and Maksym Radziwiłł. "The Fyodorov-Hiary-Keating Conjecture. II". In: *arXiv preprint arXiv:2307.00982* (2023).

References II

[5] Louis-Pierre Arguin, Guillaume Dubach, and Lisa Hartung. "Maxima of a Random Model of the Riemann zeta function over Intervals of Varying Length". In: *Annales de l'Institut Henri Poincare (B) Probabilites et statistiques*. Vol. 60. 1. Institut Henri Poincaré. 2024, pp. 588–611.

[6] Christine Chang. "Hybrid Statistics of a Random Model of zeta over Intervals of Varying Length". In: *arXiv preprint arXiv:2404.08575* (2024).

[7] Yan V Fyodorov, Ghaith A Hiary, and Jonathan P Keating. "Freezing Transition, Characteristic Polynomials of Random Matrices, and the Riemann zeta function". In: *Physical review letters* 108.17 (2012), p. 170601.

[8] Yan V Fyodorov and Jonathan P Keating. "Freezing Transitions and Extreme Values: Random Matrix Theory, and Disordered Landscapes". In: *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 372.2007 (2014), p. 20120503.

[9] Adam J Harper. "On the Partition function of the Riemann zeta function, and the Fyodorov–Hiary–Keating Conjecture". In: *arXiv preprint arXiv:1906.05783* (2019).