0

Geometry of the Spinning String

- From the electron to Spin Geometry
  - * Dirac Operator
  - * Spinor Bundles

  key: Clifford algebras
- From Feynman diagrams to pairs of pants
  - * Reduction to basic building blocks

  key: Fusion.
- The Spinor bundle on loop Space
- ... And beyond   * FFT   * 2-VB's.

1

A square root of the Laplacian

- Laplacian: $\Delta : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$,
  $\psi \longmapsto \partial_x^2\psi + \partial_y^2\psi + \partial_z^2\psi$.

Not relativistic: $i\partial_t \psi = \Delta\psi$

- Goal: A first order operator: $D^2 = \Delta$.
  $*$ $D\psi = 0 \Rightarrow \Delta\psi = 0$
- Ansatz: $D\psi = \gamma_x\partial_x\psi + \gamma_y\partial_y\psi + \gamma_z\partial_z\psi$.

$D^2 = \Delta \iff \underbrace{\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}}_{\text{These define Clifford algebras!}}$

$\underline{\text{Def}}^n$ The Clifford Algebra $Cl(\mathbb{R}^d)$ is the algebra freely generated by $\mathbb{R}^d$ subject to $vw + wv = 2\langle v,w\rangle \mathbb{1}$ $v,w \in \mathbb{R}^d$

If $\{e_i\}$ standard basis for $\mathbb{R}^d$, set
$\mathbb{R}^d \hookrightarrow Cl(\mathbb{R}^d)$
$e_i \longmapsto \underline{\gamma_i}$

$\boxed{D = \sum_i \gamma_i \partial_i}$ Dirac Operator

To make sense of $\gamma_i\psi$, pick a Rep^n $Cl(\mathbb{R}^d) \circlearrowright F$, and let $\psi \in C^\infty(\mathbb{R}^d, F)$.

"If you want to understand a physical theory, look at its symmetry group."

$\underline{\text{Def}}^n$ $Spin(d) = \left\{ a \in Cl(\mathbb{R}^d) \;\middle|\; \begin{array}{l} a \text{ is "unitary"}, a^*a = \mathbb{1} \\ a \text{ is of even degree} \end{array} \right\}$

$\langle a\psi, a\psi\rangle = \langle \psi, a^*a\psi\rangle = \langle\psi,\psi\rangle$

$\underline{\text{Theorem}}$ $Spin(d)$ is a Lie group, moreover it's the second step in the Whitehead tower of $O(d)$

$O(d) \leftarrow SO(d) \leftarrow Spin(d) \leftarrow \dots$
disconnected   connected   simply connected

2

Feynman diagrams

Extremely powerful tool to do calculations in QFT.

Goal: Calculate the probability of some "process"

Q: Two electrons go in, how do they come out?

A: Draw pictures!

QED rules

electron propagator	p	$iS_0(p)$
photon propagator	p	$-iD_{\mu\nu}(p)$
Interaction vertex		$ie_0\gamma^\mu$

Two electrons in, two out:

Find all "fillings" that are allowed by the rules, and write down the resulting formulas. (There are subleties)

There are infinitely many diagrams but the more loops a diagram has, the smaller its contribution will be.

Every picture represents a process, no matter how complicated the picture, it can always be built from elementary pieces.

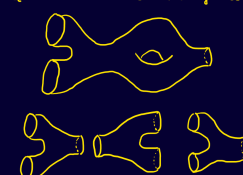
Strings

Tubes instead of lines

No longer need vertices!

pair of pants.

Q: What are the building blocks?



We can do better!

A: The fundamental Blocks are:

Fusion

3

The Spinning String

Spacetime: A nice manifold M, ($\mathbb{R}^d$ is nice enough)

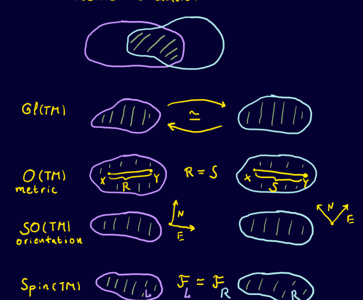
	Point Particle	String
Configuration Space	M	$LM = C^\infty(S^1, M)$
Linearization	$\mathbb{R}^d$ "without basis"	$C^\infty(S^1, \mathbb{R}^d)$ $V = \Gamma(S^1, S \otimes \mathbb{R}^d)$
γ's	$Cl(\mathbb{R}^d)$ $Cl(TM)$	$Cl(V)$ $Cl(TLM)$
Wave functions	$C^\infty(M, F)$ $\Gamma(S)$	$C^\infty(LM, F_V)$ $\Gamma(S_{LM})$
Dirac operator	$\gamma^i\partial_i$ D	Dirac-Ramond ??
Vertex	$ie_0\gamma^\mu$	$\simeq$ Fusion isomorphism $\mu : F_V \boxtimes_A F_V \longrightarrow F_V$ Waldorf-K. arXiv:1905.00222

4

Spin & String Structures

- M a d-dimensional manifold

$Gl(d) \leftarrow O(d) \leftarrow SO(d) \leftarrow Spin(d) \leftarrow String(d) \leftarrow \dots$
metric   orientation



How about $String(TM)$?

Killingback: $String(TM) \rightsquigarrow$ "$Spin(LTM)$"

Waldorf: $String(TM) \longleftrightarrow$ Fusive Spin (LTM)

Pressley-Segal (Waldorf-K.): $\underbrace{\left\{\begin{array}{l} Spin(d) \longleftrightarrow Cl(\mathbb{R}^d) \circlearrowright F \\ LSpin(d) \longleftrightarrow Cl(V) \circlearrowright F_V \end{array}\right\}}_{\text{Spinor Bundle on loop space}}$

Waldorf-K. arXiv:2009.00333

5

The Spinor Bundle on Loop Space

If $\gamma \in LM$. Replace

- $V = \Gamma(S^1, S \otimes \mathbb{R}^d)$ by $V_\gamma = \Gamma(S^1, S \otimes \gamma^* TM)$
- $Cl(V)$ by $Cl(V_\gamma)$
- F_V by $F_{V_\gamma} = F_\gamma$

Do this in such a way that F_γ depends smoothly on $\gamma \in LM$ $S = \coprod_{\gamma \in LM} F_\gamma$

Next incorporate the fusion $F_V \boxtimes_A F_V \to F_V$:

$F_{\gamma_1} \boxtimes_\beta F_{\gamma_2} \xrightarrow{\sim} F_{\gamma_3}$

Waldorf-K.
arXiv:2012.08142