Moments of symmetric-square L-functions

Dmitry Frolenkov
joint work with Olga Balkanova

Steklov Mathematical institute

4 February 2025

Symmetric-square L-function

Symmetric-square L-function attached to Hecke-Maass form is defined by

$$L(\mathrm{sym}^2 u_j, s) := \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_j(n^2)}{n^s}, \quad \Re s > 1.$$

It satisfies the functional equation:

$$L_\infty(s, t_j) L(\mathrm{sym}^2 u_j, s) = L_\infty(1-s, t_j) L(\mathrm{sym}^2 u_j, 1-s),$$

where

$$L_\infty(s, t_j) = \pi^{-3s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+2it_j}{2}\right) \Gamma\left(\frac{s-2it_j}{2}\right).$$

The analytic conductor of $L(\mathrm{sym}^2\, u_j, 1/2+2it)$ is equal to

$$Q := (1+|t|)(1+|t+t_j|)(1+|t-t_j|).$$

Therefore, when $t < (1-\delta)t_j$ the convexity estimate is:

$$|L(\mathrm{sym}^2\, u_j, 1/2+2it)| \ll Q^{1/4+\epsilon} \ll (1+|t|)^{1/4+\epsilon} |t_j|^{1/2+\epsilon}.$$

The second moment of $L(\text{sym}^2 u_j, 1/2)$ in short intervals

$$\sum_{T<t_j\leqslant T+G} \alpha_j |L(\text{sym}^2 u_j, 1/2)|^2 \ll T^{1+\epsilon} G. \tag{1}$$

- 2015 Lam proved (1) for $G > T^{1/3+\epsilon}$.
- 2016 Tang and Xu proved (1) for $G > T^{1/3+\epsilon}$.
- 2020 Khan and Young proved (1) for $G > T^{1/5+\epsilon}$.

The second moment of $L(\mathrm{sym}^2 u_j, 1/2+it)$ in short intervals

$$\sum_{T<t_j\leqslant T+G} \alpha_j |L(\mathrm{sym}^2 u_j, 1/2+it)|^2 \ll T^{1+\epsilon} G. \tag{2}$$

- 2020 Khan and Young proved (2) for $\frac{T^{3/2+\delta}}{G^{3/2}} \leqslant t \leqslant (2-\delta)T$.
- 2024 Balkanova and F. proved (2) for $G \gg \max\left(\frac{t^{2/3}}{T^{1/3}}, \frac{T}{t}\right) T^{\epsilon}$ and $t \ll T^{6/7-\epsilon}$.

Subconvexity for $L(\mathrm{sym}^2 u_j, 1/2+it)$

$$|L(\mathrm{sym}^2 u_j, 1/2+2it)| \ll (1+|t|)^{1/4+\epsilon}|t_j|^{1/2+\epsilon}.$$

Corollary (Khan-Young)

For any $0<\delta<2$ *and* $|t_j|^{6/7+\delta}<t<(2-\delta)|t_j|$ *one has*

$$|L(\mathrm{sym}^2 u_j, 1/2+it)| \ll |t_j|^{1+\epsilon}t^{-1/3}.$$

Corollary (Balkanova-F.)

One has

$$|L(\mathrm{sym}^2 u_j, 1/2+it)| \ll \frac{|t_j|^{1+\epsilon}}{\sqrt{t}} \quad \textit{for} \quad |t_j|^{2/3+\epsilon} \ll t \le |t_j|^{4/5},$$

$$|L(\mathrm{sym}^2 u_j, 1/2+it)| \ll (|t_j|t)^{1/3}|t_j|^{\epsilon} \quad \textit{for} \quad |t_j|^{4/5} \le t \ll |t_j|^{6/7-\epsilon}.$$

Other subconvexity estimates

Let F-be a self-dual Hecke-Maass cusp form for $Gl(3,\mathbb{Z})$ (that is a symmetric-square lift from GL_2) and let

$$L(F,1/2+it) \ll (1+|t|)^{\vartheta+\epsilon}.$$

Li (2011): $\vartheta = 11/16$, McKee-Sun-Ye (2015): $\vartheta = 2/3$, Nunes (2017): $\vartheta = 5/8$, Lin-Nunes-Qi (2022): $\vartheta = 3/5$, Pal (2023): $\vartheta = 129/217$, Dasgupta-Leung-Young (2024): $\vartheta = 4/7$.
Dasaratharaman-Munshi (2023):

$$L(\mathrm{sym}^2 f,1/2+it) \ll p^{1/2+\epsilon} t^{3/4-1/12+\epsilon}.$$

Ganguly-Humphries-Lin-Nunes (2024): let χ be a primitive character of conductor q and $(q_1, q/q_1) = 1$. Then

$$L(F\otimes\chi,1/2+it) \ll (q(1+|t|))^{3/5+\epsilon}\left(1+\frac{q^{2/5}}{q_1^{1/2}(1+|t|)^{1/10}}+\frac{q_1^{1/8}}{q^{1/10}(1+|t|)^{1/10}}\right).$$

Proof of Khan and Young

- apply an approximate functional equation for each of two L-functions;
- using the Kuznetsov trace formula obtain sums of the following shape: $\sum_{m,n,c} \frac{S(m^2,n^2,c)}{m^{1/2+iU} n^{1/2-iU} c} H\left(\frac{4\pi mn}{c}\right)$;
- evaluate the integral $H(x)$;
- split the sum over m and n in AP modulo c, apply the Poisson summation formula, getting $\sum_{k,l,c} \frac{1}{c^2} T(k,l,c) I(k,l,c)$, where $T(k,l,c)$ is an exponential sum and $I(k,l,c)$ is a double integral;
- evaluate $T(k,l,c)$ and $I(k,l,c)$, split variables k, l and c via the Mellin inversion;
- study the obtained multiple Dircihlet series and transform them into Dirichlet L-functions of quadratic characters;
- apply the functional equation to one of these L-functions (this can de viewed as a third application of the Poisson formula), after some computations obtain the second moment of Dirichlet L-functions of quadratic characters;
- estimate this moment using Heath-Brown's large sieve for quadratic characters.

Zagier L-series

The following L-series was independently introduced by Cohen, Zagier and Kuznetsov:

$$\mathcal{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\rho_q(n)}{q^s}, \quad \Re s > 1, \quad n \in \mathbb{Z},$$

where $\rho_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}$. Zagier(1976): the function $\mathcal{L}_n(s)$ can be meromorphically continued to the whole complex plane and the completed L–function

$$\mathcal{L}_n^*(s) = (\pi/|n|)^{-s/2}\Gamma(s/2 + 1/4 - \operatorname{sgn} n/4)\mathcal{L}_n(s)$$

satisfies the functional equation $\mathcal{L}_n^*(s) = \mathcal{L}_n^*(1-s)$. Furthermore,

- $\mathcal{L}_n(s) = 0$ if $n \equiv 2, 3 \pmod 4$;
- $\mathcal{L}_0(s) = \zeta(2s-1)$;
- For a fundamental discriminant D one has

$$\mathcal{L}_D(s) = L(s, \chi_D) = \sum_{n \geq 1} \frac{\chi_D(n)}{n^s},$$

  where $\chi_D = \left(\frac{D}{n}\right)$ – is a Kronecker symbol.

A reciprocity type formula for the first moment $L(\mathrm{sym}^2 u_j, 1/2)$

$$M_1(l,1/2;h) = MT(l;h) + CT(l;h) + ET(l;h) + S_1(l;h) + S_2(l;h),$$

$$S_1(l;h) = \sum_{n=1}^{2l-1} \frac{\mathcal{L}_{n^2-4l^2}(1/2)}{\sqrt{2\pi n}} I\left(\frac{n}{l}\right), \quad S_2(l;h) = \sum_{n=2l+1}^{\infty} \frac{\mathcal{L}_{n^2-4l^2}(1/2)}{\sqrt{2\pi n}} I\left(\frac{n}{l}\right),$$

where for $x \geq 2$

$$I(x) := \frac{2^{3/2} i}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{rh(r)}{\cosh(\pi r)} \left(\frac{2}{x}\right)^{2ir} \frac{\Gamma(1/4+ir)\Gamma(3/4+ir)}{\Gamma(1+2ir)} \times \sin\left(\pi(1/4-ir)\right) {}_2F_1\left(1/4+ir, 3/4+ir, 1+2ir; \frac{4}{x^2}\right) dr,$$

and for $0 < x < 2$

$$I(x) := \frac{2i}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{rh(r)}{\cosh(\pi r)} x^{1/2} \frac{\Gamma(1/4+ir)\Gamma(1/4-ir)}{\Gamma(1/2)} \times \cos\left(\pi(1/4+ir)\right) {}_2F_1\left(1/4+ir, 1/4-ir, 1/2; \frac{x^2}{4}\right) dr.$$

Our proof

- apply an approximate functional equation to one of $L(\mathrm{sym}^2 u_j, 1/2)$, thus reducing the problem to the first twisted moment;
- use a reciprocity type formula for the first moment (which is proved using the Kuznetsov trace formula and one Poisson summation), getting $\sum_{n,m} \mathcal{L}_{n^2-4m^2}(1/2)\frac{I(n/m)}{\sqrt{nm}}$;
- make the change of variables: $n-2m=q, \quad n+2m=r$;
- note that the function $I(n/m)$ is negligible unless $q \ll T/G^2$ and $qG^2 \ll r \ll T$. Performing one more change of variables: $r = l/q$, and rewriting the congruence condition $l \equiv 0 \pmod{q}$ via additive harmonics, we obtain
$$\sum_{q \ll T/G^2} \sum_{c|q} \sum_{\substack{0 \leqslant a < c \\ (a,c)=1}} \sum_{l \ll qT} \mathcal{L}_l(1/2) e\left(\frac{al}{q}\right) F(q,l);$$
- apply the Voronoi summation formula for the sum over l;

Our proof (continued)

$$\sum_{q \ll T/G^2} \sum_{c|q} \sum_{\substack{0 \leqslant a < c \\ (a,c)=1}} \sum_{l \ll qT} \mathcal{L}_l(1/2) e\left(\frac{al}{q}\right) F(q,l)$$

- apply Voronoi summation formula for the sum over l;
- evaluating the sum over a, we obtain either a Gauss sum or a generalized Kloosterman sum of half-integral weight. In both cases, changing the order of summation over c and q, evaluating the sum over c, we obtain the Dirichlet L-function of quadratic character;
- since Zagier L-series can also be written as Dirichlet L-functions of quadratic characters, we obtain the second moment of Dirichlet L-functions of quadratic characters;
- estimate this moment using Heath-Brown's large-sieve for quadratic characters.

Maass forms of half-integral weight

Let $\Gamma_0(4) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2, \mathbf{Z}), \quad 4|c \right\}.$

For $M \in \Gamma_0(4)$ and $k = 1/2$ we define

$$\left(f\big|_k M\right)(z) := \left(\frac{cz+d}{|cz+d|}\right)^{-k} f(Mz), \quad \Delta_k := \Delta - iyk\frac{\partial}{\partial x}.$$

Modularity condition:

$$\left(f\big|_k M\right)(z) = \vartheta(M) f(z), \quad \vartheta(M) := \bar{\epsilon_d}\left(\frac{c}{d}\right),$$

where $\epsilon_d = 1$ if $d \equiv 1 \pmod 4$ and $\epsilon_d = i$ if $d \equiv -1 \pmod 4$, here $\left(\frac{c}{d}\right)$ is the extended Jacobi symbol which coincides with the classical one if $0 < d \equiv 1 \pmod 2$, and

$$\left(\frac{c}{d}\right) = \begin{cases} \frac{c}{|c|}\left(\frac{c}{-d}\right), & c \neq 0 \\ 1 \quad d = \pm 1, & c = 0 \\ 0 \quad d \neq \pm 1, & c = 0. \end{cases}$$

Fourier-Whittaker expansion

Let f be a Maass form of weight $1/2$ and Laplace eigenvalue $\lambda = 1/4 - \rho^2$. One has:

$$f(z) = A_0(y) + \sum_{n \neq 0} a_n W_{(\operatorname{sgn} n)\frac{1}{2},\rho}(4\pi|n|y)e(2\pi inx), \quad z = x + iy,$$

where $W_{\nu,\mu}(y)$ –Whittaker function,

$$A_0(y) = \begin{cases} a_0 y^{1/2+\rho} + b_0 y^{1/2-\rho} & \rho \neq 0, \\ a_0 y^{1/2} + b_0 y^{1/2} \log y & \rho = 0. \end{cases}$$

Duke–Iwaniec(1990): proved Voronoi's formula for holomorphic cusp forms of half-integral weight, level $N \equiv 0 \pmod 4$ in case when $c \equiv 0 \pmod N$.
Bykovskii (2000): proved Voronoi's formula in case when $A_0(y) = 0$ and $c \equiv 0 \pmod N$.

Eisenstein series of half-integral weight

Let $\Gamma_\infty := \left\{ \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, m \in \mathbb{Z} \right\}$. Consider the linear combination of Eisenstein series of weight $1/2$ for the group $\Gamma_0(4)$ at the cusp ∞ and 0 :

$$E_\infty(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \frac{\Im(\gamma z)^s}{\bar{\epsilon_d} \left(\frac{c}{d}\right) (cz+d)^{1/2}}, \; E_0(z;s) = z^{-1/4} E_\infty(-1/(4z);s),$$

$$f(z;s) := \zeta(4s-1) \left(E_\infty(z;s;1/2) + \frac{1+i}{4^s} E_0(z;s;1/2) \right).$$

The following Fourier-Whittaker expansion takes place:

$$f(z;1/2+\rho) = \zeta(1+4\rho) y^{1/2+\rho} + \frac{\sqrt{\pi}\Gamma(2\rho)\zeta(4\rho)}{4^{2\rho}\Gamma(1/2+2\rho)} y^{1/2-\rho}$$
$$+ \sum_{n \neq 0} \frac{\pi^\rho \Gamma\left(\frac{1}{2}+\rho-\frac{\operatorname{sgn} n}{4}\right) \mathcal{L}_n\left(\frac{1}{2}+2\rho\right)}{2^{1+2\rho}\Gamma\left(\frac{1}{2}+2\rho\right)|n|^{1/2-\rho}} W_{\operatorname{sgn} n/4,\rho}(4\pi|n|y)\, e(nx). \quad (3)$$

Voronoi's formula

Define the Mellin transform of function ϕ as follows

$$\phi^+(s) := \int_0^\infty \phi(y) y^{s-1} dy, \quad \phi^-(s) := \int_0^\infty \phi(-y) y^{s-1} dy.$$

Let

$$R^\pm(a, x, s) := a\phi^\pm(s) \frac{\pi^s x^{-2s}}{\Gamma(s \pm k/2)},$$

$$\mathcal{R}_f(x, \rho) := R^+(a_0, x, 1/2 + \rho) + R^+(b_0, x, 1/2 - \rho) \\ + R^-(a_0, x, 1/2 + \rho) + R^-(b_0, x, 1/2 - \rho),$$

$$\Phi^{(\pm,\pm)}(x) := \frac{\cos \pi(k/2 \mp \rho)}{\sin(2\pi\rho)} \sqrt{x} J_{-2\rho}(2\sqrt{x}) - \frac{\cos \pi(k/2 \pm \rho)}{\sin(2\pi\rho)} \sqrt{x} J_{2\rho}(2\sqrt{x}),$$

$$\Phi^{(\pm,\mp)}(x) := \frac{2\sqrt{x} K_{2\rho}(2\sqrt{y})}{\Gamma(1/2 \mp k/2 - \rho)\Gamma(1/2 \mp k/2 + \rho)}.$$

Voronoi formula

$$\widehat{\phi}(y) := \int_0^\infty \left(\frac{\phi(x)}{x} \Phi^{(+,+)}(xy) + \frac{\phi(-x)}{x} \Phi^{(+,-)}(xy) \right) dx,$$

$$\widehat{\phi}(-y) := \int_0^\infty \left(\frac{\phi(x)}{x} \Phi^{(-,+)}(xy) + \frac{\phi(-x)}{x} \Phi^{(-,-)}(xy) \right) dx.$$

For $c \equiv 0 \pmod 4$, $ad \equiv 1 \pmod 4$ and $M_2 := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$

$$\sum_{n \neq 0} \phi(n) a_n e\left(\frac{an}{c}\right) = \vartheta_k(M_2) i^k \mathcal{R}_f(c, \rho) + \vartheta_k(M_2) i^k \sum_{n \neq 0} \widehat{\phi}\left(\frac{4\pi^2 n}{c^2}\right) a_n e\left(-\frac{dn}{c}\right).$$

Hypergeometric function I

$$ {}_2F_1\left(a, b, c; z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{j=1}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)j!} z^j. $$

Balkanova-F.(2024). Let $0 < z < 1$, $r \to \infty$ and $r^{-1+\delta} < \alpha < r^{-\delta}$ then one has

$$ {}_2F_1\left(1/4 + ir(1-\alpha), 3/4 + ir(1-\alpha), 1+2ir; z\right) = $$
$$ = \frac{\exp\left(2irl_1(\alpha, z)\right)}{(1-(1-\alpha^2)z)^{1/4}} \left(1 + \sum_{j=1}^{N} \frac{c_j(\alpha, r)}{(\alpha r)^j}\right) + O((\alpha r)^{-N-1}), $$

where

$$ l_1(\alpha, z) = \log 2 - \alpha \log(1+\alpha) - \log(1+\sqrt{1-(1-\alpha^2)z}) + \alpha \log(\alpha + \sqrt{1-(1-\alpha^2)z}). $$

Hypergeometric function I. Saddle point method.

$$ {}_2F_1\left(1/4+ir(1-\alpha),3/4+ir(1-\alpha),1+2ir;z\right)= $$
$$ =\frac{\Gamma(1+2ir)}{\Gamma\left(3/4+ir(1-\alpha)\right)\Gamma\left(1/4+ir(1+\alpha)\right)}\int_0^1\frac{y^{-1/4}\exp(irf(\alpha,y))}{(1-y)^{3/4}(1-zy)^{1/4}}dy, $$
$$ f(\alpha,y)=(1-\alpha)\log y+(1+\alpha)\log(1-y)-(1-\alpha)\log(1-zy). $$

the saddle-point (a solution of $\frac{\partial}{\partial y}f(\alpha,y)=0$) is $y_{\pm}=\frac{1\pm\sqrt{1-(1-\alpha^2)z}}{(1+\alpha)z}$. If $z\to 1$ and $\alpha\to 0$ both y_+ and y_- tends to the end points and coalesce if $z=1,\alpha=0$.

$$ {}_2F_1\left(1/4+ir(1-\alpha),3/4+ir(1-\alpha),1+2ir;z\right)= $$
$$ =\frac{(1-z)^{-1/4-ir(1-\alpha)}(1+Y)^{-2ir}\Gamma(1+2ir)}{\Gamma\left(1/2+2ir\alpha\right)\Gamma\left(1/2+2ir(1-\alpha)\right)}\int_0^{\infty}e^{2ir(-x+\alpha\log q(x))}\frac{dx}{\sqrt{q(x)}}, $$

$$ q(x)=\left(1-e^{-x}\right)\left((Y+1)e^{x}-Y\right),\quad Y=\frac{1-\sqrt{1-z}}{2\sqrt{1-z}}. $$

Hypergeometric function I. Temme's variant of the saddle point method.

Let $w := 2ir, \lambda := \alpha w = 2it$ then

$$\int_0^\infty e^{-w(x-\alpha\log q(x))}\frac{dx}{\sqrt{q(x)}} = \int_0^\infty q(x)^{\lambda-1}e^{-wx}\sqrt{q(x)}dx.$$

We perform the change of variables $x-\alpha\log q(x) = t-\alpha\log t + A(\alpha)$, which moves the point $x=0$ to $t=0$, the point $x=x_0$ to $t=\alpha$ and $x=+\infty$ to $t=+\infty$.

$$\int_0^\infty e^{-w(x-\alpha\log q(x))}\frac{dx}{\sqrt{q(x)}} = e^{-wA(\alpha)}\int_0^\infty e^{-w(t-\alpha\log t)}f(t)\frac{dt}{t},\ f(t) = \frac{t}{\sqrt{q(x)}}\frac{dx}{dt}$$

$$A(\alpha) = x_0-\alpha\log q(x_0)-\alpha+\alpha\log\alpha.$$

$$f(t) = f(\alpha)+(f(t)-f(\alpha)),\quad \widetilde{f}_1(t) = t\left(\frac{f(t)-f(\alpha)}{t-\alpha}\right)'.$$

$$\int_0^\infty e^{-w(t-\alpha\log t)}f(t)\frac{dt}{t} = f(\alpha)\frac{\Gamma(\lambda)}{w^\lambda}+\frac{1}{z}\int_0^\infty e^{-w(t-\alpha\log t)}\widetilde{f}_1(t)\frac{dt}{t}.$$

Hypergeometric function II

$$F_2(r,\alpha,y) := \frac{\Gamma(1/4+ir(1-\alpha))\Gamma(1/4-ir(1+\alpha))}{\Gamma(1/2)}\times$$
$$\times\, {}_2F_1\left(1/4+ir(1-\alpha), 1/4-ir(1+\alpha), 1/2; y\right).$$

Balkanova-F.(2024). Let $0<y<1$, $r\to\infty$ and $r^{1-\delta}\ll\alpha\ll r^{-\delta}$, then one has

$$F_2(r,\alpha,y)\sim e^{irl_2(\alpha,y)}\frac{2^{3/2}\pi e^{-\pi r\alpha}}{(2r\alpha)^{1/3}}\left(\frac{\alpha^2\hat{\zeta}(y)}{y-1+\alpha^2}\right)^{1/4} Ai(-(2r\alpha)^{2/3}\hat{\zeta}(y)),$$

$$\hat{\zeta}(y)=\begin{cases} -\left(3\mathbf{a}_0(\alpha,y)/(2\alpha)\right)^{2/3}, & \text{if} \quad y<1-\alpha^2 \\ \left(3\mathbf{a}_1(\alpha,y)/(2\alpha)\right)^{2/3}, & \text{if} \quad y>1-\alpha^2, \end{cases}$$

$$\mathbf{a}_0(\alpha,y)=\frac{\pi(1-\alpha)}{2}-\arctan\frac{\sqrt{y}}{\sqrt{1-\alpha^2-y}}+\alpha\arctan\frac{\alpha\sqrt{y}}{\sqrt{1-\alpha^2-y}},$$

$$\mathbf{a}_1(\alpha,y)=\alpha\log\left(\alpha\sqrt{y}+\sqrt{y-1+\alpha^2}\right)-\log\left(\sqrt{y}+\sqrt{y-1+\alpha^2}\right)-$$
$$-\alpha\log\sqrt{1-y}+(1-\alpha)\log\sqrt{1-\alpha^2},$$

Hypergeometric function II

$$\mathbf{a}_0(\alpha, y) = \frac{(1-\alpha^2)(1-\alpha^2-y)^{3/2}}{3\alpha^2 y^{3/2}}\left(1 + O\left(\frac{1-\alpha^2-y}{\alpha^2}\right)\right),$$

$$\mathbf{a}_1(\alpha, y) = \frac{(y-1+\alpha^2)^{3/2}}{3\alpha^2\sqrt{1-\alpha^2}}\left(1 + O\left(\frac{y-1+\alpha^2}{\alpha^2}\right)\right).$$

If $0 < y < 1-\alpha^2 - \frac{\alpha^{4/3}}{r^{2/3-\delta}}$ one has

$$F_2(r, \alpha, y) \ll \frac{e^{-\pi r\alpha} e^{-2r\mathbf{a}_0(\alpha,y)}}{\sqrt{r}(1-\alpha^2-y)^{1/4}} \ll \frac{e^{-\pi r\alpha}}{r^A}.$$

If $1-\alpha^2 + \frac{\alpha^{4/3}}{r^{2/3-\delta}} < y < 1$ one has

$$F_2(r, \alpha, y) \sim 2\sqrt{\pi} e^{irl_2(\alpha,y)} e^{-\pi r\alpha} \frac{\cos\left(2r\mathbf{a}_1(\alpha, y) - \pi/4\right)}{\sqrt{r}(y-1+\alpha^2)^{1/4}},$$

$$l_2(\alpha, y) = \alpha\log(1-y) - 2\alpha\log r + (1-\alpha)\log(1-\alpha) - (1+\alpha)\log(1+\alpha) + 2\alpha.$$

$$Y_r(x) = x^{1/4}(1-x)^{1/2-ir\alpha}{}_2F_1\left(1/4+ir(1-\alpha), 1/4-ir(1+\alpha), 1/2; x\right)$$

$$Y_r''(x) = \left((2r)^2 f(\alpha,x) + g(\alpha,x)\right) Y_r(x),$$

$$f(\alpha,x) = \frac{1-\alpha^2-x}{4x(1-x)^2}, \quad g(\alpha,x) = -\frac{1}{4(1-x)^2} - \frac{3}{16x^2(1-x)}.$$

This differential equation has a double pole at $x=1$ and a turning point at $x = 1-\alpha^2$, which coalesce if $\alpha \to 0$. Asymptotic expansions of solutions of such equations were investigated by Boyd-Dunster (1986) and Dunster (1990),(2013). It turns out that the asymptotic expansion is given in terms of $K_{2ir\alpha}(\cdot)$ and $\tilde{I}_{2ir\alpha}(\cdot)$ Bessel functions. These functions itself can be approximated by Airy functions.

Symmetric-square: level aspect

First moment: Sun (2014): N-squarefree , Balkanova (2025): prime power level.
Iwaniec-Michel (2001): For a squarefree N one has

$$\sum_{f\in H^*_{2k}(N)}^{h} |L(\operatorname{sym}^2 f, 1/2+it)|^2 \ll (1+|t|)^8 N^{1+\epsilon}.$$

Blomer (2008): For a prime N, a real primitive real character $\chi(modN)$ one has

$$\sum_{f\in H^*_{2k}(N,\chi)}^{h} \lambda_f(l^2)|L(\operatorname{sym}^2 f, 1/2)|^2 = MT(l,N) + O\left(\frac{(lN)^\epsilon}{N^{1/4}} + \frac{l(lN)^\epsilon}{N^{1/2}}\right).$$

Symmetric-square: weight aspect

First moment: Lau (2002) $ET(1,k) \ll k^{-0.008}$, Fomenko (2003-2005) $ET(1,k) \ll k^{-1/2}$, Khan (2007) $ET(1,k) \ll k^{-1/20}$, Sun (2013) $ET(1,k) \ll k^{-1/2}$, Ng Ming-Ho (2016) $ET(l,k) \ll lk^{-1/2}$, Liu (2016) $ET(1,k) \ll k^{-B}$, Balkanova-F. (2016) $ET(1,k) \ll e^{-ck}$, $ET(l,k) \ll l^{5/6+\epsilon}k^{-1/2}$.
Khan (2010)

$$\frac{1}{K}\sum_{k\equiv 0 (mod\, 2)} h\left(\frac{k-1}{K}\right)\sum_{f\in H_{2k}}^{h} \lambda_f(l^2)L(\mathrm{sym}^2 f, s)^2 = MT(l,k) + O(l^{1/2}K^{\epsilon}).$$

Das-Khan (2018)

$$\frac{1}{K}\sum_{k\equiv 0 (mod\, 2)} h\left(\frac{k-1}{K}\right)\sum_{f\in H_{2k}}^{h} L(\mathrm{sym}^3 f, s)^3 = MT(l,k) + O(K^{-1/2+\epsilon}).$$

Thank you for attention!