Finding mirrors to Fano quiver flag zero loci.

Plan:
1) Motivation: why quiver flag zero loci? why mirrors?
2) Computing the A-side: the quantum period
3) Finding mirrors: the B-side. (Laurent polynomials) via a generalisation of the Gelfand–Cetlin toric degenerate

§ Motivation: Fano classification & mirror symmetry

Fano variety: smooth variety $/\mathbb{C}$ s.t. $-K_X$ is ample.

There are only finitely many n-dim Fano varieties up to deformation (Kollár–Miyaoka–Mori).

Classification: dim 1 & 2 classical

dim 3 105 (Iskovskih, Mori–Mukai)

$\Rightarrow$ All dim ≤ 3 Fano varieties can be constructed as $Z(s) \subseteq V /\!\!/_{G}$, where $s \in \Gamma(\underbrace{E \times V^{ss}/G}_{\text{representation theoretic v.b.}})$ and E is a representation of G.

In fact, they are either

1) toric complete intersection ($V/\!\!/G$ is a toric variety, G is abelian)
   - well understood

2) quiver flag zero loci ($V/\!\!/G$ is a quiver flag variety, G is non-Abelian)
   - topic for today

Theme: reduce type 2) to type 1) via

→ abelianisation

→ toric degenerations.

Quiver flag varieties:

- Q: acyclic quiver with a unique source $Q_0 = \{0, 1, \ldots, \rho\}$
- $r = (1, r_1, \ldots, r_\rho)$, $G = \prod_{i=1}^{\rho} Gl(r_i)$, $\theta = (1, r_1, \ldots, r_\rho)$

$$V/\!/G = M_\theta(Q, r) = \left. \bigoplus_{a \in Q_1} Hom(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}}) /\!/_\theta G \right]$$ quiver flag variety

eg $\underset{1}{\cdot} \xrightarrow{n} \underset{r_1}{\cdot} \rightarrow \underset{r_2}{\cdot} \rightarrow \cdots \rightarrow \cdot\, r_\rho \rightsquigarrow V/\!/G = Fl(n, r_1, \ldots, r_\rho)$

$M_\theta(Q, r)$ is a

- smooth projective variety
- fine moduli space
- MDS

} Craw

- $M_\theta(Q, r) = Z(s) \subseteq \prod_{i=1}^{\rho} Gr(\tilde{s}_i, r_i)$ $\quad s \in \Gamma(E)$ giving incidence conditions

$S_i \hookrightarrow Q_i$

$\tilde{s}_i$ = # paths $0 \to i$

Quiver flag zero loci: $W_i := Q_i|_{M_\theta(Q, r)}$.

$Z(s)$, $s \in \Gamma(\bigoplus S^{\alpha_1^i} W_1 \otimes \cdots \otimes S^{\alpha_\rho^i} W_\rho)$

Dim 4: start by classifying Fano varieties of type 1 or 2.

Restrict ambient space $\dim V/\!/G \leq 8$. → can enumerate all dim 4 Fano subvarieties of type 1) ((Coates-Kasprzyk-Prince) and

2) Coates – K – Kasprzyk

(CCGGKT...) → 141 new Fano 4folds.

Program: Use mirror symmetry to classify Fano varieties.

Conjecturally (Kasprzyk – Tveiten):

n dim Fano varieties / deformation $\longleftrightarrow$ rigid maximally mutable Laurent polynomials $f \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}]$ / mutation

X is mirror to f if

$$\underbrace{\hat{G}_X}_{\text{regularised quantum period}}(t) = \underbrace{\pi_f}_{\text{classical period}}(t)$$

$$G_X(t) = \sum_{i=0}^{\infty} \underbrace{a_i}_{} t^i$$

↳ genus 0 Gromov–Witten invariant

If X is a tci, Givental's mirror thm gives a closed form.

$$\pi_f(t) = \sum_{i=0}^{\infty} \mathrm{const}(f^i)\, t^i$$

<u>Mutations</u> $f \xrightarrow{\text{mutation}} f'$

Compositions of

a) $(\mathbb{C}^*)^n \xrightarrow{\varphi_{A \in GL(n,\mathbb{Z})}} (\mathbb{C}^*)^n$ $\qquad f' = \varphi_A^*(f)$

b) Let $h \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}]$. Define

$$(\mathbb{C}^*)^n \overset{\varphi_h}{\dashrightarrow} (\mathbb{C}^*)^n \qquad z_i \to z_i \quad i<n$$

$$z_n \to h \cdot z_n \quad i=n$$

$$f' = \varphi_h^*(f)$$

f is <u>compatible</u> with this mutation if the result, f', is a Laurent polynomial.

<u>Rigid max mutable Laurent polynomials</u>

Let $P = \mathrm{Newt}(f)$. f is given by a choice of coefficients for the lattice points of P.

A Laurent polynomial is <u>rigid maximally</u> mutable if it is compatible with a maximal set of mutations, and the coefficients are uniquely determined by this property.

<u>Evidence for the conjecture:</u>

1) X tci $\xrightarrow[\text{Hori-Vafa mirror + Przyjalkowski method}]{} f$ $\}$ verified that it is rigid max mutable in search.

requires technical condition: existence of a convex partition

2) Conjectural mirrors found for 99/141 Fano quiver flag zero loci $(K-)$

2

$$\underbrace{X}_{\substack{\text{quiver}\\ \text{flag zero locus}}} \subseteq M_\theta(Q, r) \longrightarrow \underbrace{Z}_{\text{"tci"}} \subseteq \underbrace{M_\nu(L(\theta), (1, \dots, 1))}_{\substack{\text{generalisation of Gelfand}\\ \text{- Cetlin toric degeneration}\\ \text{of flag varieties}}}$$

Przyjalkowski method (when convex partition exists)

$$f', \quad P = \mathrm{Newt}(f) \xrightarrow[\substack{\text{fix coefficients}\\ \text{to be rigid}\\ \text{max mutable}\\ \text{(Kasprzyk code)}}]{} f$$

WRONG

1 Compute 20 terms of $G_X(t)$, $\pi_f(t)$ and compare

& Computing $G_X(t)$: Abelianisation.

$$\begin{array}{ccc} E_G & & E_T \\ \downarrow & & \downarrow \\ Z(s) \subseteq V/\!\!/_\theta G & & V/\!\!/_\theta T \supseteq Z(\tilde{s}) \end{array}$$

$G \supseteq T$ max torus

Different dimensions → but turns out $V/\!\!/_\theta T$ is very useful

Results on: cohomology, quantum cohomology, Mori-wall-and-chamber structure, I functions

Even nicer when $V/\!\!/_\theta G = M_\theta(Q, r)$

Q (Ciocan-Fontanine–Kim–Sabbah, K., Webb)

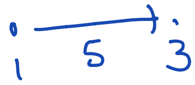
toric quiver flag variety Q^{ab}

Thm: The quantum period of $M_\theta(Q, r)$ can be computed via the quantum period of $M_\theta(Q^{ab}, (1, \dots, 1))$.

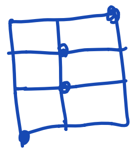
§ Laurent polynomial mirrors.

Hard to find a toric degeneration of $Z \subseteq M_\theta(Q, r)$ directly: instead try to generalise known constructions for flag varieties.

Ladder quivers



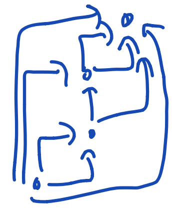
$Gr(5,3)$



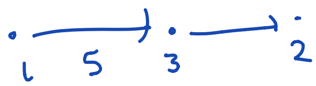
+ vertices

Orient

$\rightarrow \uparrow$



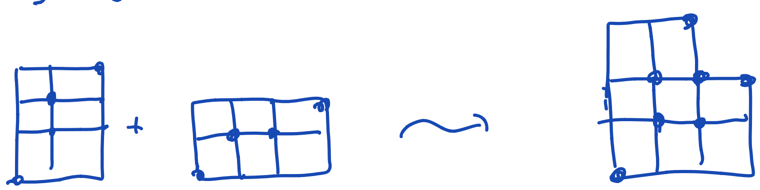
$L(Q)$

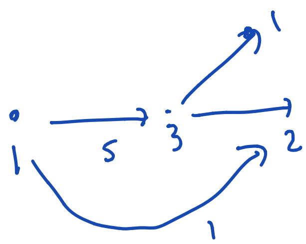


$Fl(5,3,2) \subseteq Gr(5,3) \times Gr(5,2)$

$(\mathbb{C}^5/V_1, \mathbb{C}^5/V_2)$

$V_2 \supseteq V_1$



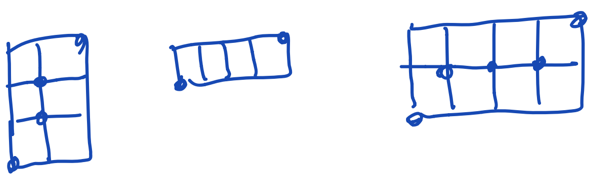


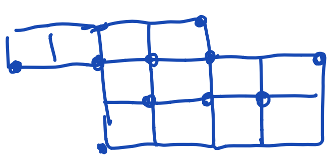
$M_6(Q,r) \subseteq Gr(5,3) \times Gr(5,1) \times Gr(6,2)$

$(\mathbb{C}^5/V_1, \mathbb{C}^5/V_2, \mathbb{C}^6/V_3)$

$V_1 \subseteq V_2$

$V_1 \oplus \mathbb{C} \subseteq V_3$





Works for any Y-shaped quiver

$Q \longrightarrow L(Q)$

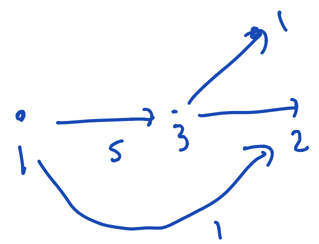
Thm (R-) There is a toric degeneration of a Fano Y-shaped quiver to $M_\nu(L(Q),(1,\dots,1))$ where ν is the Fano stability condition.

Pf: via finding a SAGBI basis of

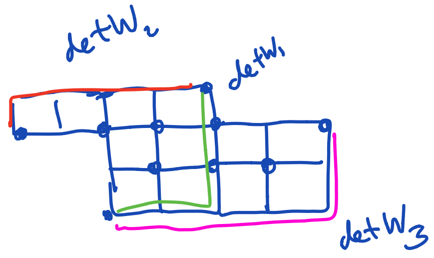
$$M_\theta(Q,r) \subseteq \prod Gr(\tilde{s}_i, r_i) \subseteq \prod \mathbb{P}^{a_i}$$

i.e. using the embedding given by the $\det W_i$.

and comparing with $M_\nu(L(Q),(1,\dots,1)) \hookrightarrow \prod \mathbb{P}^{a_i}$



path $\longrightarrow$ Weil divisor in $M_\nu(L(Q),(1,\dots,1))$.



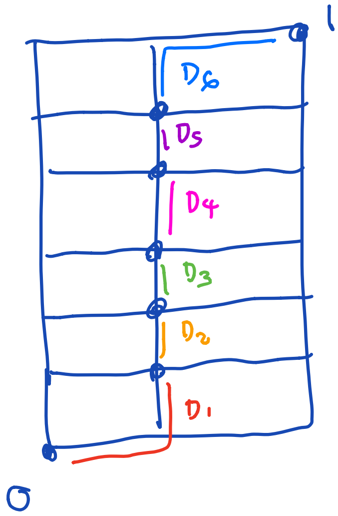
Mirrors for subvarieties:

$Z \subseteq M_\theta(Q,r)$: if Z is a ci, then Z degenerates to a tci in $M_\nu(Q,r)$.

(used by Prince to find mirrors to ci in flag varieties)

General zero loci: use combinatorics of $L(Q)$.

$Gr(8,6) \subseteq \mathbb{P}^{\binom{8}{6}-1}$



Any path $0 \to 1 \rightsquigarrow \mathcal{O}(1) \Leftarrow \det W_1$

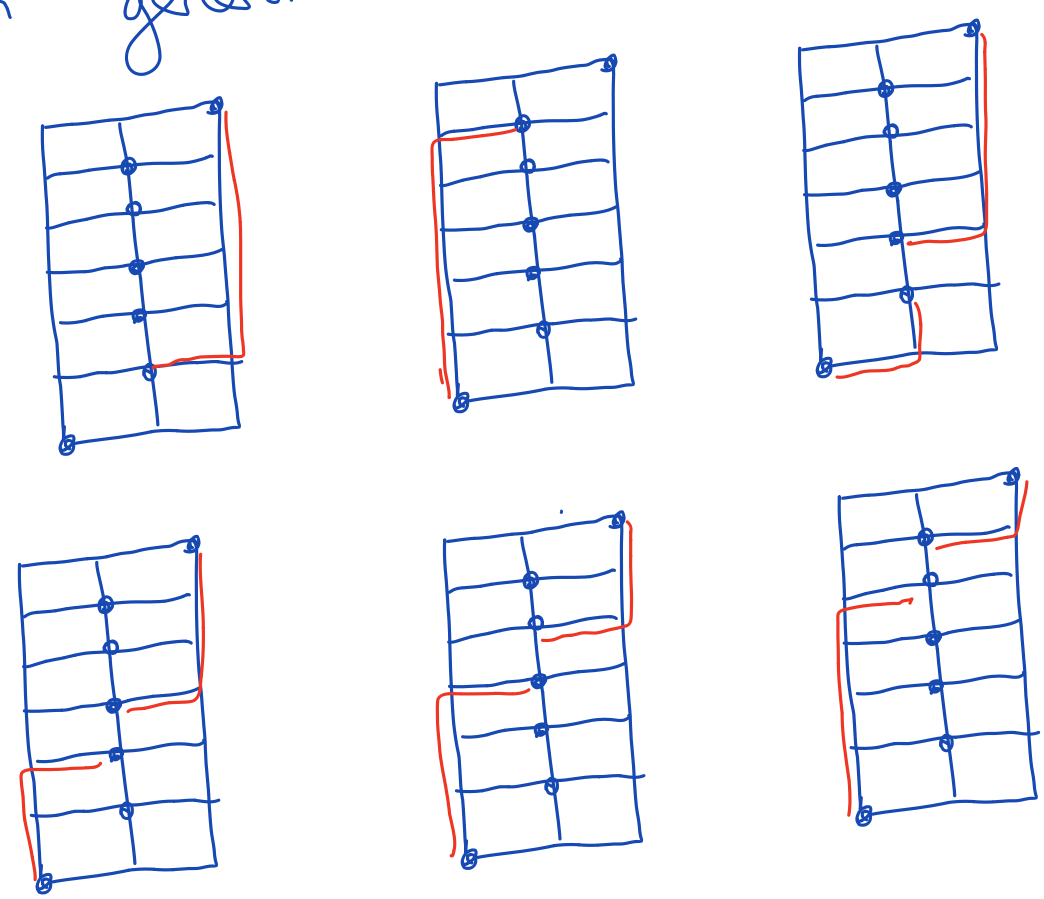
$$\mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \otimes \mathcal{O}(D_3) \otimes \mathcal{O}(D_4) \otimes \mathcal{O}(D_5) \otimes \mathcal{O}(D_6)$$

$$= \mathcal{O}(1)|_{M_\theta(L(Q),(1,\dots,1))} \to \det W_1$$

$$\mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \oplus \mathcal{O}(D_3) \oplus \mathcal{O}(D_4) \oplus \mathcal{O}(D_5) \oplus \mathcal{O}(D_6)$$

$$\sim W_1$$

Can generalise to $S^\alpha W_1$ → eg $\Lambda^5 W_1$ on $Gr(8,6)$



→ produce a Laurent polynomial mirror to $Z(s) \subseteq Gr(8,6)$, $s \in \Gamma(\det W_1 \oplus \det W_1 \oplus \Lambda^5 W_1)$