p-torsion of Jacobians for unramified $\mathbb{Z}/p\mathbb{Z}$-covers of curves (joint with Bryden Cais)

March 27, 2023

Douglas Ulmer

University of Arizona

Number Theory and Combinatorics Seminar

University of Lethbridge

Outline

p-torsion of Jacobians for unramified $\mathbb{Z}/p\mathbb{Z}$-covers of curves

1. p-torsion group schemes
2. Dieudonné theory and de Rham cohomology
3. E–O stratification of $\mathcal{A}_g$ and the motivating question
4. Previous results
5. New results
6. Making calculations

Group schemes

Group schemes are schemes (algebraic varieties) with group structure.

Group schemes

Group schemes are schemes (algebraic varieties) with group structure. E.g., for a commutative ring R:

$$\mathbb{G}_m(R) = R^\times \qquad \mathbb{G}_a(R) = R_+$$

and

$$\mathrm{GL}_n(R) = n \times n \text{ invertible matrices over } R.$$

Group schemes

Group schemes are schemes (algebraic varieties) with group structure. E.g., for a commutative ring R:

$$\mathbb{G}_m(R) = R^\times \qquad \mathbb{G}_a(R) = R_+$$

and

$$\mathrm{GL}_n(R) = n \times n \text{ invertible matrices over } R.$$

Other examples include elliptic curves and abelian varieties.

p-torsion group schemes

If k is a field of characteristic $p > 0$ and R is a k-algebra,

$$\mu_p(R) = \{a \in R \mid a^p = 1\} \text{ with multiplication as group law}$$

p-torsion group schemes

If k is a field of characteristic $p > 0$ and R is a k-algebra,

$$\mu_p(R) = \{a \in R \mid a^p = 1\} \text{ with multiplication as group law}$$

$$\alpha_p(R) = \{a \in R \mid a^p = 0\} \text{ with addition as group law}$$

p-torsion group schemes

If k is a field of characteristic $p > 0$ and R is a k-algebra,

$$\mu_p(R) = \{a \in R \mid a^p = 1\} \text{ with multiplication as group law}$$

$$\alpha_p(R) = \{a \in R \mid a^p = 0\} \text{ with addition as group law}$$

$$\mathbb{Z}/p\mathbb{Z}(R) = (\mathbb{Z}/p\mathbb{Z})^{\pi_0(\operatorname{Spec} R)} = \operatorname{Mor}(\operatorname{Spec} R, \mathbb{Z}/p\mathbb{Z}).$$

p-torsion group schemes

If k is a field and E is an elliptic curve over k, then (by definition)

$$E[p](R) = \{x \in E(R) \mid px = 0\}$$

p-torsion group schemes

If k is a field and E is an elliptic curve over k, then (by definition)

$$E[p](R) = \{x \in E(R) \mid px = 0\}$$

If k is algebraically closed (or even perfect) of char $p > 0$, then there are two possibilities for $E[p]$:

$$E[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \quad \text{"ordinary"}$$

p-torsion group schemes

If k is a field and E is an elliptic curve over k, then (by definition)

$$E[p](R) = \{x \in E(R) \mid px = 0\}$$

If k is algebraically closed (or even perfect) of char $p > 0$, then there are two possibilities for $E[p]$:

$$E[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \quad \text{``ordinary''}$$

or there is a non-split exact sequence

$$0 \to \alpha_p \to E[p] \to \alpha_p \to 0$$

"supersingular".

p-torsion group schemes

Still assuming k is algebraically closed of char $p > 0$, if A is an abelian variety of dimension g over k, then there are exactly 2^g possibilities for $A[p]$.

p-torsion group schemes

Still assuming k is algebraically closed of char $p > 0$, if A is an abelian variety of dimension g over k, then there are exactly 2^g possibilities for $A[p]$. The most common ("ordinary") is

$$A[p] \cong E_{ord}[p]^g \cong (\mathbb{Z}/p\mathbb{Z})^g \oplus (\mu_p)^g$$

p-torsion group schemes

Still assuming k is algebraically closed of char $p > 0$, if A is an abelian variety of dimension g over k, then there are exactly 2^g possibilities for $A[p]$. The most common ("ordinary") is

$$A[p] \cong E_{ord}[p]^g \cong (\mathbb{Z}/p\mathbb{Z})^g \oplus (\mu_p)^g$$

and the most special ("superspecial") is

$$A[p] \cong E_{ss}[p]^g$$

p-torsion group schemes

Still assuming k is algebraically closed of char $p > 0$, if A is an abelian variety of dimension g over k, then there are exactly 2^g possibilities for $A[p]$. The most common ("ordinary") is

$$A[p] \cong E_{ord}[p]^g \cong (\mathbb{Z}/p\mathbb{Z})^g \oplus (\mu_p)^g$$

and the most special ("superspecial") is

$$A[p] \cong E_{ss}[p]^g$$

The isomorphism class of $A[p]$ is called its "Ekedahl–Oort type". It's reasonable to think of it as some kind of Lie algebra.

More background on p-torsion group schemes

A group scheme $\mathcal{G}$ over k killed by p has endomorphisms F and V with $FV = VF = 0$.

$\mathcal{G}$ is étale if $V = 0$, F bijective (e.g., $\mathbb{Z}/p\mathbb{Z}$).

More background on p-torsion group schemes

A group scheme $\mathcal{G}$ over k killed by p has endomorphisms F and V with $FV = VF = 0$.

$\mathcal{G}$ is étale if $V = 0$, F bijective (e.g., $\mathbb{Z}/p\mathbb{Z}$).

$\mathcal{G}$ is multiplicative if $F = 0$, V bijective (e.g., μ_p).

More background on p-torsion group schemes

A group scheme $\mathcal{G}$ over k killed by p has endomorphisms F and V with $FV = VF = 0$.

$\mathcal{G}$ is étale if $V = 0$, F bijective (e.g., $\mathbb{Z}/p\mathbb{Z}$).

$\mathcal{G}$ is multiplicative if $F = 0$, V bijective (e.g., μ_p).

$\mathcal{G}$ is local-local if F and V are nilpotent (e.g., α_p or $E_{ss}[p]$).

More background on p-torsion group schemes

A group scheme $\mathcal{G}$ over k killed by p has endomorphisms F and V with $FV = VF = 0$.

$\mathcal{G}$ is étale if $V = 0$, F bijective (e.g., $\mathbb{Z}/p\mathbb{Z}$).

$\mathcal{G}$ is multiplicative if $F = 0$, V bijective (e.g., μ_p).

$\mathcal{G}$ is local-local if F and V are nilpotent (e.g., α_p or $E_{ss}[p]$).

Every $\mathcal{G}$ decomposes canonically into a direct sum of étale, multiplicative and l-l subgroups.

p-torsion of Jacobians

Let X be a curve of genus g over k, let J_X be its Jacobian, and let $J_X[p]$ be the p-torsion of J_X. This is a group scheme of order p^{2g}.

p-torsion of Jacobians

Let X be a curve of genus g over k, let J_X be its Jacobian, and let $J_X[p]$ be the p-torsion of J_X. This is a group scheme of order p^{2g}.

The p-rank (or "f-number") of J_X is the largest integer f so that

$$(\mathbb{Z}/p\mathbb{Z})^f \hookrightarrow J_X[p]$$

(or equivalently $(\mu_p)^f \hookrightarrow J_X[p]$).

p-torsion of Jacobians

Let X be a curve of genus g over k, let J_X be its Jacobian, and let $J_X[p]$ be the p-torsion of J_X. This is a group scheme of order p^{2g}.

The p-rank (or "f-number") of J_X is the largest integer f so that

$$(\mathbb{Z}/p\mathbb{Z})^f \hookrightarrow J_X[p]$$

(or equivalently $(\mu_p)^f \hookrightarrow J_X[p]$).

The a-number of J_X is the largest integer so that

$$(\alpha_p)^f \hookrightarrow J_X[p].$$

p-torsion of Jacobians

Let X be a curve of genus g over k, let J_X be its Jacobian, and let $J_X[p]$ be the p-torsion of J_X. This is a group scheme of order p^{2g}.

The p-rank (or "f-number") of J_X is the largest integer f so that

$$(\mathbb{Z}/p\mathbb{Z})^f \hookrightarrow J_X[p]$$

(or equivalently $(\mu_p)^f \hookrightarrow J_X[p]$).

The a-number of J_X is the largest integer so that

$$(\alpha_p)^f \hookrightarrow J_X[p].$$

We have $0 \le f \le g$ and $0 \le a \le g$ and $1 \le a + f \le g$.

Example: $X = E$ ordinary $\Rightarrow f = 1,\ a = 0$

$X = E$ supersingular $\Rightarrow f = 0,\ a = 1.$

Dieudonné theory

Let $\mathbb{D}$ be the k-algebra generated by symbols F and V with relations

$$FV = VF = 0, \qquad F\alpha = \alpha^p F, \qquad \alpha V = V\alpha^p$$

for all $\alpha \in k$. (This is the Dieudonné ring over k.)

Dieudonné theory

Let $\mathbb{D}$ be the k-algebra generated by symbols F and V with relations

$$FV = VF = 0, \qquad F\alpha = \alpha^p F, \qquad \alpha V = V\alpha^p$$

for all $\alpha \in k$. (This is the Dieudonné ring over k.)

There is a contravariant equivalence of categories between finite groups schemes over k killed by p and finite-dimensional $\mathbb{D}$-modules. Write $M(G)$ for the module associated to a group scheme G.

Examples:

$$
\begin{aligned}
M(\mathbb{Z}/p\mathbb{Z}) &\cong \mathbb{D}/(F-1, V) \cong k \quad \text{with } F = id, V = 0,\\
M(\mu_p) &\cong \mathbb{D}/(F, V-1) \cong k \quad \text{with } F = 0, V = id,\\
M(\alpha_p) &\cong \mathbb{D}/(F, V) \cong k \quad \text{with } F = V = 0.
\end{aligned}
$$

Dieudonné theory

Examples:

$$M(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{D}/(F-1, V) \cong k \quad \text{with } F = id, V = 0,$$
$$M(\mu_p) \cong \mathbb{D}/(F, V-1) \cong k \quad \text{with } F = 0, V = id,$$
$$M(\alpha_p) \cong \mathbb{D}/(F, V) \cong k \quad \text{with } F = V = 0.$$

If E is a supersingular elliptic curve,

$$M(E[p]) \cong \mathbb{D}/(F-V) \cong k^2.$$

Dieudonné theory

For a curve X, the module $M(J_X[p])$ is a "self-dual BT_1 module," meaning that it admits a non-degenerate, alternating pairing, and it satisfies $\ker F = \operatorname{Im} V$ and $\ker V = \operatorname{Im} F$.

Dieudonné theory

For a curve X, the module $M(J_X[p])$ is a "self-dual BT_1 module," meaning that it admits a non-degenerate, alternating pairing, and it satisfies $\ker F = \operatorname{Im} V$ and $\ker V = \operatorname{Im} F$.

There are several nice classifications of self-dual BT_1-modules in terms of words on the alphabet $\{f, v\}$, certain sequences of integers (E-O structures), Weyl group elements, ...

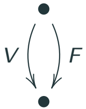
Dieudonné theory

A self-dual BT_1 module is described by a multi-set of "primitive cyclic words" in $\{f, v\}$ which is invariant under exchanging f and v. E.g.,

$$M(E_{ord}[p]) \leftrightarrow (f), (v)$$

and

$$M(E_{ss}[p]) \leftrightarrow (fv)$$



Dieudonné theory

Self-dual B=BT_1 modules of dimension $2g$ are also described by E-O structures, namely sequences

$$n_0 = 0 \leq n_1 \leq \cdots \leq n_g$$

where $n_i \leq n_{i+1} \leq n_i + 1$. There are 2^g of these. E.g.,

$$M(E_{ord}[p]) \leftrightarrow [1]$$

$$M(E_{ss}[p]) \leftrightarrow [0]$$

Oda's theorem

Oda proved that $M(J_X[p])$ is the first de Rham cohomology of X.

We'll just recall a concrete description of $H^1_{dR}(X)$ with its $\mathbb{D}$-module structure.

Oda's theorem

Oda proved that $M(J_X[p])$ is the first de Rham cohomology of X.

We'll just recall a concrete description of $H^1_{dR}(X)$ with its $\mathbb{D}$-module structure.

If X is covered by two affine open subsets U_1 and U_2, then

$$H^1_{dR}(X) \cong \frac{\{(\omega_1, \omega_2, f_{12}) |\ df_{12} = \omega_1 - \omega_2\}}{\{(dg_1, dg_2, g_1 - g_2)\}}.$$

Oda's theorem

Oda proved that $M(J_X[p])$ is the first de Rham cohomology of X.

We'll just recall a concrete description of $H^1_{dR}(X)$ with its $\mathbb{D}$-module structure.

If X is covered by two affine open subsets U_1 and U_2, then

$$H^1_{dR}(X) \cong \frac{\{(\omega_1, \omega_2, f_{12}) |\ df_{12} = \omega_1 - \omega_2\}}{\{(dg_1, dg_2, g_1 - g_2)\}}.$$

We define

$$F(\omega_1, \omega_2, f_{12}) = (0, 0, f_{12}^p) \qquad V(\omega_1, \omega_2, f_{12}) = (\mathcal{C}\omega_1, \mathcal{C}\omega_2, 0)$$

where $\mathcal{C}$ is the Cartier operator.

Oda's theorem

Oda proved that $M(J_X[p])$ is the first de Rham cohomology of X.

We'll just recall a concrete description of $H^1_{dR}(X)$ with its $\mathbb{D}$-module structure.

If X is covered by two affine open subsets U_1 and U_2, then

$$H^1_{dR}(X) \cong \frac{\{(\omega_1, \omega_2, f_{12}) |\ df_{12} = \omega_1 - \omega_2\}}{\{(dg_1, dg_2, g_1 - g_2)\}}.$$

We define

$$F(\omega_1, \omega_2, f_{12}) = (0, 0, f_{12}^p) \qquad V(\omega_1, \omega_2, f_{12}) = (\mathcal{C}\omega_1, \mathcal{C}\omega_2, 0)$$

where $\mathcal{C}$ is the Cartier operator.

These are things that can be explicitly calculated on a machine (as Bryden and I have done a lot)!

Motivating question

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties over k. Then $\mathcal{A}_g$ has a nice stratification by E–O types (the E–O stratification).

Motivating question

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties over k. Then $\mathcal{A}_g$ has a nice stratification by E–O types (the E–O stratification).

Let $\mathcal{M}_g$ be the moduli space of curves of genus g. We have a closed immersion

$$\mathcal{M}_g \hookrightarrow \mathcal{A}_g \qquad X \mapsto J_X$$

and it is of great interest to study how the image of $\mathcal{M}_g$ behaves with respect to the E–O stratification.

Motivating question

Ample evidence shows that that $\mathcal{M}_g$ is not transverse to this stratification.

Motivating question

Ample evidence shows that that $\mathcal{M}_g$ is not transverse to this stratification.

For example, there are curves of every genus over $\mathbb{F}_2$ with $J_X[p]$ more special than predicted by dimension considerations.

Motivating question

Ample evidence shows that that $\mathcal{M}_g$ is not transverse to this stratification.

For example, there are curves of every genus over $\mathbb{F}_2$ with $J_X[p]$ more special than predicted by dimension considerations.

Understanding this failure motivates our main question: What are the possibilities for $J_X[p]$ for curves X?

Motivating question

Ample evidence shows that that $\mathcal{M}_g$ is not transverse to this stratification.

For example, there are curves of every genus over $\mathbb{F}_2$ with $J_X[p]$ more special than predicted by dimension considerations.

Understanding this failure motivates our main question: What are the possibilities for $J_X[p]$ for curves X?

A theme of a lot of contemporary research is to construct curves X where $J_X[p]$ is interesting, e.g., more special than expected.

Motivating question

See Pries-Ulmer NYJM 2022 for a survey of E–O structures and many examples. In Proc. AMS 2022, we showed that every self-dual BT_1 group scheme appears as a direct factor of $J_X[p]$ for an explicit curve X (usually a Fermat curve).

Motivating question

See Pries-Ulmer NYJM 2022 for a survey of E–O structures and many examples. In Proc. AMS 2022, we showed that every self-dual BT_1 group scheme appears as a direct factor of $J_X[p]$ for an explicit curve X (usually a Fermat curve).

Note that this says that every BT_1 appears as a *direct factor* of some $J_X[p]$, but maybe not as $J_X[p]$ itself.

p-torsion and unramified $\mathbb{Z}/p\mathbb{Z}$-covers

Let X be a nice curve over k and let $Y \to X$ be an unramified Galois cover with an isomorphism $\mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$ (also called an Artin–Schreier cover). What are the relationships between $J_X[p]$ and $J_Y[p]$?

p-torsion and unramified $\mathbb{Z}/p\mathbb{Z}$-covers

Let X be a nice curve over k and let $Y \to X$ be an unramified Galois cover with an isomorphism $\mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$ (also called an Artin–Schreier cover). What are the relationships between $J_X[p]$ and $J_Y[p]$?

$$\text{Deuring–Shafarevich:} \quad f_Y - 1 = p(f_X - 1)$$

p-torsion and unramified $\mathbb{Z}/p\mathbb{Z}$-covers

Let X be a nice curve over k and let $Y \to X$ be an unramified Galois cover with an isomorphism $\mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$ (also called an Artin–Schreier cover). What are the relationships between $J_X[p]$ and $J_Y[p]$?

$$\text{Deuring–Shafarevich:} \quad f_Y - 1 = p(f_X - 1)$$

$$\text{Booher–Cais:} \quad a_X \leq a_Y \leq p a_X$$

p-torsion and unramified $\mathbb{Z}/p\mathbb{Z}$-covers

Let X be a nice curve over k and let $Y \to X$ be an unramified Galois cover with an isomorphism $\mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$ (also called an Artin–Schreier cover). What are the relationships between $J_X[p]$ and $J_Y[p]$?

$$\text{Deuring–Shafarevich:} \quad f_Y - 1 = p(f_X - 1)$$

$$\text{Booher–Cais:} \quad a_X \leq a_Y \leq p a_X$$

Our goal today is to refine, extend, and give more structure to these results.

G-modules

By Oda, $H^1_{dR}(X)$ and $H^1_{dR}(Y)$ are the $\mathbb{D}$-modules associated to $J_X[p]$ and $J_Y[p]$, and our main question is “how they are related?”

G-modules

By Oda, $H^1_{dR}(X)$ and $H^1_{dR}(Y)$ are the $\mathbb{D}$-modules associated to $J_X[p]$ and $J_Y[p]$, and our main question is "how they are related?"

There is one more structure on $H^1_{dR}(Y)$, namely the action of $G = \mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$.

G-modules

By Oda, $H^1_{dR}(X)$ and $H^1_{dR}(Y)$ are the $\mathbb{D}$-modules associated to $J_X[p]$ and $J_Y[p]$, and our main question is "how they are related?"

There is one more structure on $H^1_{dR}(Y)$, namely the action of $G = \mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$.

Let $R = k[G]$, so that

$$R = k[G] \cong k[g]/(g^p - 1) = k[g]/(g-1)^p \cong k[\delta]/(\delta^p).$$

G-modules

By Oda, $H^1_{dR}(X)$ and $H^1_{dR}(Y)$ are the $\mathbb{D}$-modules associated to $J_X[p]$ and $J_Y[p]$, and our main question is "how they are related?"

There is one more structure on $H^1_{dR}(Y)$, namely the action of $G = \mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$.

Let $R = k[G]$, so that

$$R = k[G] \cong k[g]/(g^p - 1) = k[g]/(g-1)^p \cong k[\delta]/(\delta^p).$$

The indecomposable finite-dimensional R-modules are

$$V_i := k[\delta]/(\delta^i) \qquad \text{for } i = 1, \ldots, p,$$

and V_p is free over R of rank 1.

Chevalley-Weil for $H^1_{dR}(Y)$

The key result is an isomorphism of $k[G]$-modules:

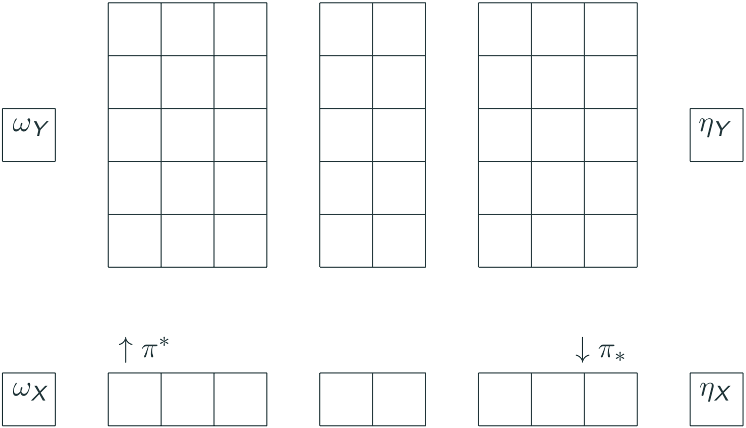
$$H^1_{dR}(Y) \cong V_1^2 \oplus V_p^{2g_X - 2}$$

Chevalley-Weil for $H^1_{dR}(Y)$

The key result is an isomorphism of $k[G]$-modules:

$$H^1_{dR}(Y) \cong V_1^2 \oplus V_p^{2g_X - 2}$$

Pictorially ($p = 5$, $g_X = 5$):



Consequences for $J_Y[p]$

Suppose $k = \overline{k}$. Then there are (self-dual BT_1) group schemes $\mathcal{G}_X$ and $\mathcal{G}_Y$ over k such that

$$J_X[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus \mathcal{G}_X,$$
$$J_Y[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus \mathcal{G}_Y,$$

Consequences for $J_Y[p]$

Suppose $k = \overline{k}$. Then there are (self-dual BT_1) group schemes $\mathcal{G}_X$ and $\mathcal{G}_Y$ over k such that

$$J_X[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus \mathcal{G}_X,$$
$$J_Y[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus \mathcal{G}_Y,$$

and there is a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_p = \mathcal{G}_Y$$

by subgroup schemes such that

$$\mathcal{G}_X \cong \mathcal{G}_i/\mathcal{G}_{i-1} \qquad \text{for } i = 1, \ldots, p.$$

Consequences for $J_Y[p]$

Suppose $k = \overline{k}$. Then there are (self-dual BT_1) group schemes $\mathcal{G}_X$ and $\mathcal{G}_Y$ over k such that

$$J_X[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus \mathcal{G}_X,$$
$$J_Y[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p \oplus \mathcal{G}_Y,$$

and there is a filtration

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_p = \mathcal{G}_Y$$

by subgroup schemes such that

$$\mathcal{G}_X \cong \mathcal{G}_i/\mathcal{G}_{i-1} \qquad \text{for } i = 1, \ldots, p.$$

So we have a filtration on $J_Y[p]$ with known associated graded, and "all" we have to do is examine extensions and reassemble $J_Y[p]$ from $J_X[p]$.

Bad news on extensions

Unfortunately, the category of BT_1 modules is very badly behaved with respect to extensions. The simples have been classified by Oort, but there is no Jordan-Holder Theorem: A given (self-dual, BT_1) group scheme G may have two filtrations with different Jordan-Holder factors.

Bad news on extensions

Unfortunately, the category of BT_1 modules is very badly behaved with respect to extensions. The simples have been classified by Oort, but there is no Jordan-Holder Theorem: A given (self-dual, BT_1) group scheme G may have two filtrations with different Jordan-Holder factors.

E.g., the module with word (f^3v^3) is a repeated extension of three copies of (fv) (a simple BT_1 module), and it is also an extension of (f^2v) by (fv^2) (both of which are simple).

Bad news on extensions

Unfortunately, the category of BT_1 modules is very badly behaved with respect to extensions. The simples have been classified by Oort, but there is no Jordan-Holder Theorem: A given (self-dual, BT_1) group scheme G may have two filtrations with different Jordan-Holder factors.

E.g., the module with word (f^3v^3) is a repeated extension of three copies of (fv) (a simple BT_1 module), and it is also an extension of (f^2v) by (fv^2) (both of which are simple).

So we have to scale back our ambitions on describing $J_Y[p]$ completely as a BT_1 module with $\mathbb{Z}/p\mathbb{Z}$ action. That said, we have some interesting results.

Applications to $J_Y[p]_{ét}$

Suppose $k = \overline{k}$. Then the Deuring–Shafarevich formula refines to an isomorphism of group schemes

$$J_Y[p]_{ét} \cong \mathbb{Z}/p\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{F}_p[G])^{f_X - 1}$$

with action of $G = \mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$.

Applications to $J_Y[p]_{ét}$

Suppose $k = \overline{k}$. Then the Deuring–Shafarevich formula refines to an isomorphism of group schemes

$$J_Y[p]_{ét} \cong \mathbb{Z}/p\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{F}_p[G])^{f_X - 1}$$

with action of $G = \mathrm{Gal}(Y/X) \cong \mathbb{Z}/p\mathbb{Z}$.

(This has been observed previously by many people and serves mostly as a reality check for us.)

Applications to $J_Y[p]_{ét}$

Now suppose k is a general perfect field and define ν_X and ν_Y by

$$|J_X[p](k)| = p^{\nu_X} \quad \text{and} \quad |J_Y[p](k)| = p^{\nu_Y}$$

So $\nu_X \leq f_X$ with equality if $k = \overline{k}$.

Applications to $J_Y[p]_{ét}$

Now suppose k is a general perfect field and define ν_X and ν_Y by

$$|J_X[p](k)| = p^{\nu_X} \quad \text{and} \quad |J_Y[p](k)| = p^{\nu_Y}$$

Then we have bounds

$$\nu_X \leq \nu_Y$$

Applications to $J_Y[p]_{ét}$

Now suppose k is a general perfect field and define ν_X and ν_Y by

$$|J_X[p](k)| = p^{\nu_X} \quad \text{and} \quad |J_Y[p](k)| = p^{\nu_Y}$$

Then we have bounds

$$\nu_X \leq \nu_Y \leq 1 + p(\nu_X - 1) \quad \text{(under splitting conditions)}$$

Applications to $J_Y[p]_{ét}$

Now suppose k is a general perfect field and define ν_X and ν_Y by

$$|J_X[p](k)| = p^{\nu_X} \quad \text{and} \quad |J_Y[p](k)| = p^{\nu_Y}$$

Then we have bounds

$$\nu_X \leq \nu_Y \leq \begin{cases} 1 + p(\nu_X - 1) & \text{under splitting conditions,} \\ p\nu_X & \text{in general} \end{cases}$$

Applications to $J_Y[p]_{ét}$

Now suppose k is a general perfect field and define ν_X and ν_Y by

$$|J_X[p](k)| = p^{\nu_X} \quad \text{and} \quad |J_Y[p](k)| = p^{\nu_Y}$$

Then we have bounds

$$\nu_X \leq \nu_Y \leq \begin{cases} 1 + p(\nu_X - 1) & \text{under splitting conditions,} \\ p\nu_X & \text{in general} \end{cases}$$

"Splitting conditions" refers to:

$$J_X[p]_{ét} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$$

Applications to $J_Y[p]_{ét}$

Continuing to assume only that k is perfect: If $f_X = 1$, then $f_Y = 1$ and $J_Y[p]_{ét} \cong \mathbb{Z}/p\mathbb{Z}$. The next case is more interesting:

Applications to $J_Y[p]_{ét}$

Continuing to assume only that k is perfect: If $f_X = 1$, then $f_Y = 1$ and $J_Y[p]_{ét} \cong \mathbb{Z}/p\mathbb{Z}$. The next case is more interesting:

Suppose that k is finite and $f_X = 2$. Then we are in one of three cases:

(1a) $f_X = f_Y = 1$

(1b) $f_X = 1 < f_Y < p + 1$

(2) $f_X = 2 \leq f_Y \leq p + 1$

Applications to $J_Y[p]_{ét}$

Continuing to assume only that k is perfect: If $f_X = 1$, then $f_Y = 1$ and $J_Y[p]_{ét} \cong \mathbb{Z}/p\mathbb{Z}$. The next case is more interesting:

Suppose that k is finite and $f_X = 2$. Then we are in one of three cases:

(1a) $f_X = f_Y = 1$

(1b) $f_X = 1 < f_Y < p + 1$

(2) $f_X = 2 \leq f_Y \leq p + 1$

Moreover, there is a presentation of $M(J_Y[p])$ by generators and relations determined just by these numerical invariants, and over an extension of k of degree dividing $p(p-1)$,

$$J_Y[p] \cong (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{F}_p[G]) .$$

Applications to $J_Y[p]_{ll}$

There is a lot to say about the local-local part, but we just state two results here.

Applications to $J_Y[p]_{ll}$

There is a lot to say about the local-local part, but we just state two results here.

Thm: Suppose that k is algebraically closed and that $f_X = g_X - 1$. (This implies that $a_X = 1$.) If $p = 2$, then a_Y is 1 or 2. If $p > 2$, then $a_Y \in \{2, 4, \ldots, p-1, p\}$. Moreover the local-local part of $J_Y[p]$ has an explicit description in terms of generators and relations depending only on a_Y.

Applications to $J_Y[p]_{ll}$

There is a lot to say about the local-local part, but we just state two results here.

Thm: Suppose that k is algebraically closed and that $f_X = g_X - 1$. (This implies that $a_X = 1$.) If $p = 2$, then a_Y is 1 or 2. If $p > 2$, then $a_Y \in \{2, 4, \ldots, p-1, p\}$. Moreover the local-local part of $J_Y[p]$ has an explicit description in terms of generators and relations depending only on a_Y.

This is a substantial refinement of Booher–Cais.

Applications to $J_Y[p]_{ll}$

Thm: Suppose that $J_X[p]_{ll}$ is superspecial, i.e., $J_X[p]_{ll} \cong E_{ss}[p]^h$ where $h = g_X - f_X$. Then the Ekedahl–Oort structure of $J_Y[p]_{ll}$ starts with h zeroes, i.e., it has the form $[0, 0, \ldots, 0, \psi_{h+1}, \ldots, \psi_{ph}]$.

Applications to $J_Y[p]_{ll}$

Thm: Suppose that $J_X[p]_{ll}$ is superspecial, i.e., $J_X[p]_{ll} \cong E_{ss}[p]^h$ where $h = g_X - f_X$. Then the Ekedahl–Oort structure of $J_Y[p]_{ll}$ starts with h zeroes, i.e., it has the form $[0, 0, \ldots, 0, \psi_{h+1}, \ldots, \psi_{ph}]$.

The theorem reduces the number of possibilities for $J_X[p]_{ll}$ from 2^{ph} to $2^{(p-1)h}$.

Variants: More freedom

When X has a k-rational point, Bryden introduced a certain enlargement of $J_Y[p]$ which is G-free with associated graded equal to p copies of $J_X[p]$.

Variants: More freedom

When X has a k-rational point, Bryden introduced a certain enlargement of $J_Y[p]$ which is G-free with associated graded equal to p copies of $J_X[p]$.

Roughly speaking, the Dieudonné module of the enlargement is

$$\mathbb{H}^1(Y, \mathcal{O}_Y(-D) \to \Omega^1_Y(D))$$

where D is the inverse image in Y of the chosen point. (This is the de Rham realization of some 1-motive).

Variants: More freedom

When X has a k-rational point, Bryden introduced a certain enlargement of $J_Y[p]$ which is G-free with associated graded equal to p copies of $J_X[p]$.

Roughly speaking, the Dieudonné module of the enlargement is

$$\mathbb{H}^1(Y, \mathcal{O}_Y(-D) \to \Omega^1_Y(D))$$

where D is the inverse image in Y of the chosen point. (This is the de Rham realization of some 1-motive).

The enlargement *does depend* in an interesting way on the choice of the point.

Variants: Ramified covers

The crucial Chevalley-Weil result is that

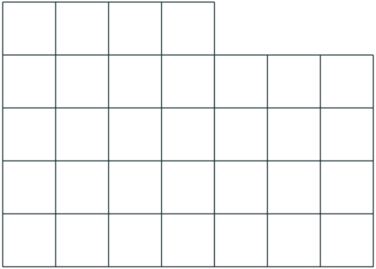
$$H^1_{dR}(Y) \cong V_p^{2g_X} \oplus V_{p-1}^r$$

Variants: Ramified covers

The crucial Chevalley-Weil result is that

$$H^1_{dR}(Y) \cong V_p^{2g_X} \oplus V_{p-1}^r$$

Pictorially ($p = 5$, $g_X = 2$, $r = 3$):



Calculations

The cover $\pi : Y \to X$ can be recovered from X and the sheaf of $\mathcal{O}_X$-algebras $\mathcal{F} := \pi_* \mathcal{O}_X$ as a global Spec.

Calculations

The cover $\pi : Y \to X$ can be recovered from X and the sheaf of $\mathcal{O}_X$-algebras $\mathcal{F} := \pi_* \mathcal{O}_X$ as a global Spec.

$\mathcal{F}$ has a p-step filtration, and it satisfies $\mathcal{F} \cong \mathrm{Sym}^{p-1} \mathcal{G}$ where $\mathcal{G} := \mathrm{Fil}^2 \mathcal{F}$.

Calculations

The cover $\pi : Y \to X$ can be recovered from X and the sheaf of $\mathcal{O}_X$-algebras $\mathcal{F} := \pi_* \mathcal{O}_X$ as a global Spec.

$\mathcal{F}$ has a p-step filtration, and it satisfies $\mathcal{F} \cong \mathrm{Sym}^{p-1} \mathcal{G}$ where $\mathcal{G} := \mathrm{Fil}^2 \mathcal{F}$.

$\mathcal{G}$ is a rank two vector bundle on X (an extension of $\mathcal{O}_X$ by itself), so is described by a class in $H^1(X, \mathcal{O}_X)$. It has a very compact description in terms of a transition function. (Need ϵ more to recover the algebra structure on $\mathcal{F}$.)

Calculations

The cover $\pi : Y \to X$ can be recovered from X and the sheaf of $\mathcal{O}_X$-algebras $\mathcal{F} := \pi_* \mathcal{O}_X$ as a global Spec.

$\mathcal{F}$ has a p-step filtration, and it satisfies $\mathcal{F} \cong \mathrm{Sym}^{p-1} \mathcal{G}$ where $\mathcal{G} := \mathrm{Fil}^2 \mathcal{F}$.

$\mathcal{G}$ is a rank two vector bundle on X (an extension of $\mathcal{O}_X$ by itself), so is described by a class in $H^1(X, \mathcal{O}_X)$. It has a very compact description in terms of a transition function. (Need ϵ more to recover the algebra structure on $\mathcal{F}$.)

The upshot is that $H^1_{dR}(Y)$ is the cohomology *on* X of the de Rham complex of X with coefficients in $\mathcal{F}$, and $\mathcal{F}$ has the compact description above.

Calculations

We now recall how to compute $H^1(X, \mathcal{O}_X)$ (with its Frobenius):

Calculations

We now recall how to compute $H^1(X, \mathcal{O}_X)$ (with its Frobenius):

Choose a "non-special" divisor D, i.e., $H^1(X, \mathcal{O}_X(D)) = 0$. Then

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)/\mathcal{O}_X \to 0$$

is an acyclic resolution of $\mathcal{O}_X$.

Calculations

We now recall how to compute $H^1(X, \mathcal{O}_X)$ (with its Frobenius):

Choose a "non-special" divisor D, i.e., $H^1(X, \mathcal{O}_X(D)) = 0$. Then

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)/\mathcal{O}_X \to 0$$

is an acyclic resolution of $\mathcal{O}_X$.

We find that

$$\frac{H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(D))} \cong H^1(X, \mathcal{O}_X).$$

Calculations

We now recall how to compute $H^1(X, \mathcal{O}_X)$ (with its Frobenius):

Choose a "non-special" divisor D, i.e., $H^1(X, \mathcal{O}_X(D)) = 0$. Then

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)/\mathcal{O}_X \to 0$$

is an acyclic resolution of $\mathcal{O}_X$.

We find that

$$\frac{H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(D))} \cong H^1(X, \mathcal{O}_X).$$

The numerator is something purely local, and the denominator is a standard Riemann-Roch space. So machine computation of the LHS is possible.

Calculations

For Frobenius, note that pD is non-special if D is, so we have isomorphisms

$$\frac{H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(D))} \cong H^1(X, \mathcal{O}_X) \cong \frac{H^0(X, \mathcal{O}_X(pD)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(pD))}.$$

Calculations

For Frobenius, note that pD is non-special if D is, so we have isomorphisms

$$\frac{H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(D))} \cong H^1(X, \mathcal{O}_X) \cong \frac{H^0(X, \mathcal{O}_X(pD)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(pD))}.$$

To compute Frobenius, take a representative principal part, raise it to the p-th power, then "reduce" it back to $\mathcal{O}_X(D)/\mathcal{O}_X$ using global sections of $\mathcal{O}_X(pD)$.

Calculations

For Frobenius, note that pD is non-special if D is, so we have isomorphisms

$$\frac{H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(D))} \cong H^1(X, \mathcal{O}_X) \cong \frac{H^0(X, \mathcal{O}_X(pD)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(pD))}.$$

To compute Frobenius, take a representative principal part, raise it to the p-th power, then "reduce" it back to $\mathcal{O}_X(D)/\mathcal{O}_X$ using global sections of $\mathcal{O}_X(pD)$.

A mild generalization of this method works to compute the hypercohomology of the complex $\mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$, i.e., $H^1_{dR}(Y)$ with its Frobenius. Recover V using the de Rham pairing.

Calculations

For Frobenius, note that pD is non-special if D is, so we have isomorphisms

$$\frac{H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(D))} \cong H^1(X, \mathcal{O}_X) \cong \frac{H^0(X, \mathcal{O}_X(pD)/\mathcal{O}_X)}{H^0(X, \mathcal{O}_X(pD))}.$$

To compute Frobenius, take a representative principal part, raise it to the p-th power, then "reduce" it back to $\mathcal{O}_X(D)/\mathcal{O}_X$ using global sections of $\mathcal{O}_X(pD)$.

A mild generalization of this method works to compute the hypercohomology of the complex $\mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$, i.e., $H^1_{dR}(Y)$ with its Frobenius. Recover V using the de Rham pairing.

All this is implemented in Magma and we used it to produce many examples and counterexamples.

Thank You