Distribution of Gaussian primes and zeros of L-functions

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Writing a prime as a sum of two squares

If $p \equiv 1 \pmod 4$, there is a unique way to write $p = a^2 + 4b^2$ with a, b integers, $a, b > 0$.

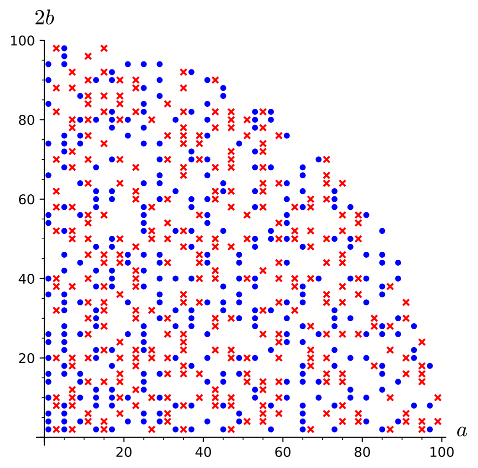
Theorem (Hecke)

We have

$$\#\{p \leq x : p = a^2 + 4b^2, |a| \equiv 1 \pmod 4\} \sim \#\{p \leq x : p = a^2 + 4b^2, |a| \equiv 3 \pmod 4\} \sim \frac{1}{4}\pi(x).$$

Is there a bias ?

Writing a prime as a sum of two squares

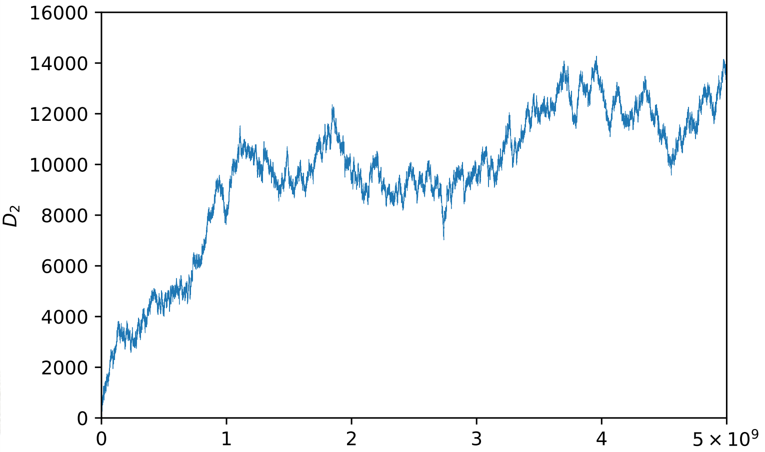


Prime numbers with $p = a^2 + 4b^2 < 10000$,
$a, b > 0$, blue dots : $a \equiv 1 \pmod 4$, red crosses : $a \equiv 3 \pmod 4$.

Is there a bias ?

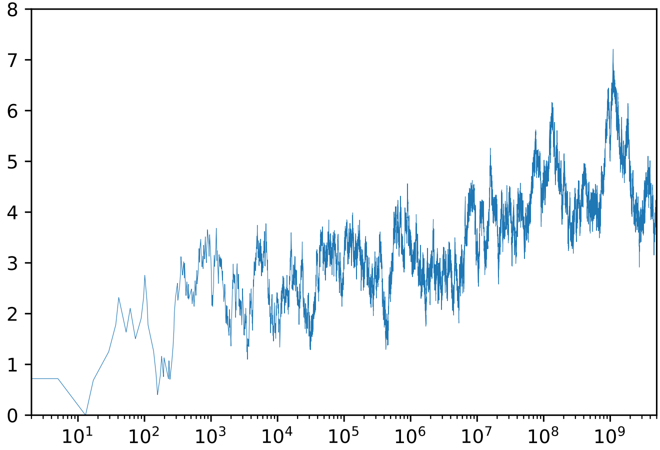
La bienveillance, avec laquelle vous avez toujours agréé mes recherches m'engage à vous présenter un nouveau résultat relatif aux nombres premiers que je viens de trouver. En cherchant l'expression limitative des fonctions qui déterminent la totalité des nombres premiers de la forme $a^2 + 4b^2$ *pour lesquels* a *est de la forme* $4n + 1$ *et ceux pour lesquels* a *est de la forme* $4n + 3$*, je suis parvenu à reconnaître que ces deux fonctions diffèrent notablement entre elles par leurs seconds termes, dont la valeur, pour les nombres* $4n + 1$ *est plus grande que celle pour les nombres* $4n + 3$ *;*

Fouvry's bias



$$\#\{p \leq x : p = a^2 + 4b^2, |a| \equiv 1 \pmod 4\} - \#\{p \leq x : p = a^2 + 4b^2, |a| \equiv 3 \pmod 4\}$$

Fouvry's bias, normalized



$$\frac{\log x}{\sqrt{x}}\Big(\#\{p \le x : p = a^2 + 4b^2, |a| \equiv 1 \pmod 4\} - \#\{p \le x : p = a^2 + 4b^2, |a| \equiv 3 \pmod 4\}\Big),$$

normalized, logarithmic scale

Interpretation

Consider the CM elliptic curve with affine equation $E : y^2 = x^3 + x$.
Write

$$a_p = p + 1 - |E(\mathbb{F}_p)| = 2\sqrt{p}\cos\tilde{\theta}_p = \sqrt{p}(\xi(\mathfrak{p}) + \xi(\overline{\mathfrak{p}})),$$

for $(p) = \mathfrak{p}\overline{\mathfrak{p}}$,
where ξ is the Hecke character on the multiplicative groups of fractional ideals of $\mathbf{Z}[i]$ modulo (4) defined by

$$\xi((\alpha)) = \begin{cases} \frac{\alpha}{|\alpha|} & \text{if } \alpha \equiv 1 \bmod (4) \\ -\frac{\alpha}{|\alpha|} & \text{if } \alpha \equiv 3 + 2i \bmod (4) \\ 0 & \text{if } (\alpha, (4)) \neq 1. \end{cases}$$

Then if $p = a^2 + 4b^2$,

$$|a| \equiv 1 \bmod 4 \quad \Leftrightarrow \quad \cos\tilde{\theta}_p > 0.$$

Explicit formula

Assuming the Generalized Riemann Hypothesis, we obtain that

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod 4}} \cos(\tilde{\theta}_p) = \frac{x^{\frac{1}{2}}}{2\log x}\left(1 - \sum_{\substack{|\gamma|\le T\\ L(\frac{1}{2}+i\gamma,\xi)=0}} \frac{e^{i\gamma\log x}}{\frac{1}{2}+i\gamma} + R_1(x,T)\right)$$

where $R_1(x,T)$ is small in average for a good choice of T.

Explicit formula – again

Assuming the Generalized Riemann Hypothesis, we obtain that

$$\sum_{\substack{p \le x \\ p \equiv 1 \pmod 4}} \cos(m\tilde{\theta}_p) = \frac{x^{\frac{1}{2}}}{2\log x} \left(-(-1)^m - \sum_{\substack{|\gamma| \le T \\ L(\frac{1}{2}+i\gamma, \xi^m)=0}} \frac{e^{i\gamma \log x}}{\frac{1}{2} + i\gamma} + R_m(x,T) \right)$$

where $R_m(x,T)$ is small in average for a good choice of T.

Summing the explicit formulas

Following an idea of Sarnak in a letter to Mazur, we have

$$\sum_{\substack{p \le x \\ p \equiv 1 \pmod 4}} \sum_{m \ge 1} C_m \cos(m\tilde{\theta}_p)$$

$$= \frac{x^{\frac{1}{2}}}{\log x} \sum_{m \ge 1} C_m \left(-\frac{(-1)^m}{2} - \frac{1}{2} \sum_{\substack{|\gamma| \le T \\ L(\frac{1}{2}+i\gamma, \xi^m)=0}} \frac{e^{i\gamma \log x}}{\frac{1}{2} + i\gamma} + R_m(x,T) \right)$$

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where $R_m(x,T)$ are small in average for a good choice of T.

We have $L(\frac{1}{2}, \xi^m) = 0$ for $m \equiv 3 \,(\mathrm{mod}\, 4)$.

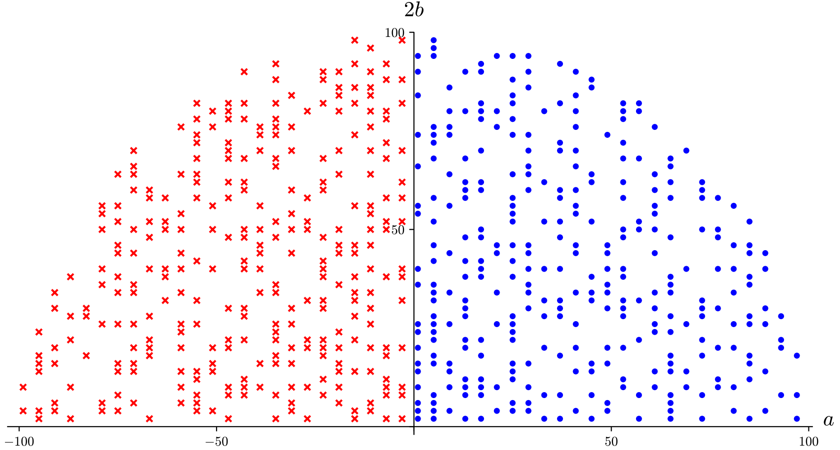
Even, 2π-periodic smooth functions

Theorem (D.)

Assume the Generalized Riemann Hypothesis, and that for all $m \in \mathbf{N}_{>0}$ we have $L(\frac{1}{2}, \xi^m) = 0$ with order 1 if and only if $m \equiv 3 \pmod 4$.
Then, for any ϕ smooth, even, 2π-periodic with $\int_0^\pi \phi = 0$ the function $y \mapsto \frac{y}{e^{y/2}} \sum_{p \le e^y} \phi(\tilde{\theta}_p)$ admits a limiting distribution with mean value equal to

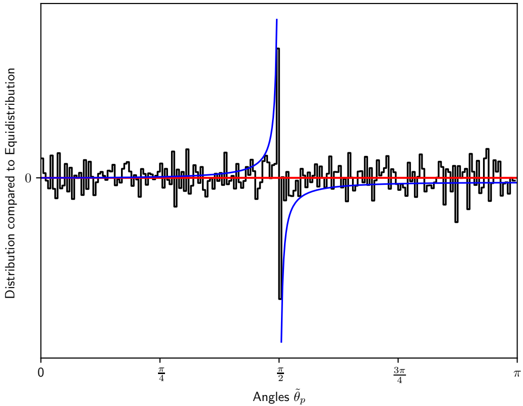
$$-\frac{\phi(0) + \phi(\pi)}{4} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{1}{2\cos(t)} \, \mathrm{d}t.$$

Modified angles



Prime numbers with $p = a^2 + 4b^2 < 10000$,
unfolded : $\sqrt{p}e^{i\tilde{\theta}_p}$ blue dots : $a \equiv 1 \pmod 4$, red crosses : $a \equiv 3 \pmod 4$.

Distribution of the angles of unfolded Gaussian primes



Distribution of the angles $\tilde{\theta}_p$ in intervals $[\frac{k\pi}{200}, \frac{(k+1)\pi}{200})$ for $p \equiv 1 \pmod 4$, $p \leq 10^7$. equidistribution, with secondary term : $\frac{1}{2\cos(t)}$

Counting zeros at $\frac{1}{2}$ in a family

Let $\mathcal{F}$ be a *family* of L-functions, and

$$\mathcal{F}(Q) = \begin{cases} \{f \in \mathcal{F} : c_f \leq Q\} \text{ or} \\ \{f \in \mathcal{F} : c_f = Q\} \text{ or} \\ \{f \in \mathcal{F} : Q \leq c_f \leq 2Q\} \text{ or ...} \end{cases}$$

with $|\mathcal{F}(Q)| \xrightarrow[Q\to\infty]{} \infty$.

Average over the family :

Definition (Average rank for a family of L-function)

Let $\mathcal{F}$ be a family of L-functions, we define

$$\mathcal{R}(\mathcal{F}(Q)) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} \operatorname{ord}_{s=\frac{1}{2}} L(s, f)$$

Counting zeros ~~at~~ close to $\frac{1}{2}$ in a family

Let $\mathcal{F}$ be a *family* of L-functions, and

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with $|\mathcal{F}(Q)| \xrightarrow[Q \to \infty]{} \infty$.

Average over the family :

Definition (One-level density for a family of L-function)

Let $\mathcal{F}$ be a family of L-functions, let ϕ be an even Schwartz function we define

$$\mathcal{D}(\mathcal{F}(Q), \phi) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} \sum_{\substack{\gamma \\ L(\frac{1}{2}+i\gamma, f)=0}} \phi(\frac{\gamma}{2\pi} \log c_f)$$

The Katz–Sarnak heuristic

Inspired by ideas of Dyson, Montgomery and Odlyzko, using function fields analogues and random matrix models,

Conjecture (Katz–Sarnak)

For any family $\mathcal{F}$, there exist a symmetry type $G(\mathcal{F})$, such that for any even Schwartz function ϕ, one has

$$\lim_{Q\to\infty} \mathcal{D}(\mathcal{F}(Q),\phi) = \int \phi(x) W(G(\mathcal{F}))(x)\,\mathrm{d}x$$

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$$\lim_{Q\to\infty} \mathcal{D}(\mathcal{F}(Q),\phi) = \int \phi(x) W(G(\mathcal{F}))(x)\,\mathrm{d}x$$
$$= \int \hat{\phi}(x) \widehat{W(G(\mathcal{F}))}(x),$$

where $\widehat{\phi}(\xi) := \int_{\mathbf{R}} \phi(x) e^{-2\pi i \xi x}\,\mathrm{d}x$.

Katz–Sarnak heuristic – symmetry types

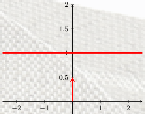
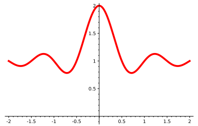
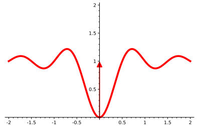
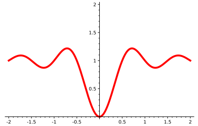
Symmetry type	$\int \phi W$	graph of W
O	$\int \phi + \frac{1}{2}\phi(0)$	
SO(even)	$\int \phi + \int \phi(x) \frac{\sin(2\pi x)}{2\pi x}\, \mathrm{d}x$	
SO(odd)	$\int \phi + \phi(0) - \int \phi(x) \frac{\sin(2\pi x)}{2\pi x}\, \mathrm{d}x$	
Sp	$\int \phi - \int \phi(x) \frac{\sin(2\pi x)}{2\pi x}\, \mathrm{d}x$	

Table – Possible symmetry types in real families

Katz–Sarnak heuristic – symmetry types

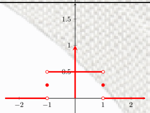
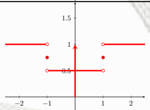
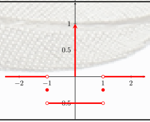
Symmetry type	$\int \hat{\phi}\hat{W}$	graph of $\hat{W}$
O	$\hat{\phi}(0) + \frac{1}{2}\phi(0)$	
$SO(\text{even})$	$\hat{\phi}(0) + \frac{1}{2}\int_{-1}^{1}\hat{\phi}$	
$SO(\text{odd})$	$\hat{\phi}(0) + \phi(0) - \frac{1}{2}\int_{-1}^{1}\hat{\phi}$	
Sp	$\hat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\hat{\phi}$	

Table – Possible symmetry types in real families

Proportion of non-vanishing

Take

$$\phi(x) = \Big(\frac{\sin(\pi\sigma x)}{\pi\sigma x}\Big)^2 \qquad \widehat{\phi}(t) = \begin{cases} \frac{\sigma - |t|}{\sigma^2} \text{ if } |t| < \sigma \\ 0 \text{ otherwise.} \end{cases}$$

Then the proportion of f in $\mathcal{F}$ such that $L(\frac{1}{2}, f) \neq 0$ is at least

$$\frac{3}{4} - \frac{1}{2\sigma} \qquad SO(\text{even})$$

$$\frac{5}{4} - \frac{1}{2\sigma} \qquad SO(\text{odd})$$

$$\frac{5}{4} - \frac{1}{2\sigma} \qquad Sp$$

Cutting the family

Let ξ_k be the primitive character inducing ξ^k,

$$\mathcal{F}^\alpha := \{L(s,\xi_k) \ : \ k \geq 1, \ k \equiv \alpha \bmod 4\}.$$

$$\mathcal{D}(K;\phi,\mathcal{F}^\alpha) := \frac{4}{K} \sum_{\substack{1 \leq k \leq K \\ k \equiv \alpha \bmod 4}} \sum_{L(\frac{1}{2}+i\gamma,\xi_k)=0} \phi\Big(\frac{\gamma \log k}{\pi}\Big). \tag{1}$$

Theorem (David–D.–Waxman)

Let ϕ be an even Schwartz function with supp$(\widehat{\phi}) \subset (-1,1)$. When α is even,

$$\mathcal{D}(K;\phi,\mathcal{F}^\alpha) = \widehat{\phi}(0) - \frac{1}{2}\int_{\mathbb{R}} \widehat{\phi}(u)\,\mathrm{d}u + O\big(\tfrac{1}{\log K}\big)$$

and when α is odd,

$$\mathcal{D}(K;\phi,\mathcal{F}^\alpha) = \widehat{\phi}(0) + \frac{1}{2}\int_{\mathbb{R}} \widehat{\phi}(u)\,\mathrm{d}u + O\big(\tfrac{1}{\log K}\big).$$

Vanishing at $\frac{1}{2}$

Corollary

Let $\alpha \in \mathbb{Z}/4\mathbb{Z}$. Then the proportion of non-vanishing in $\mathcal{F}_d^\alpha$ is at least

$$\lim_{K\to\infty} \frac{4}{K} \# \left\{1 \leq k \leq K,\, k \equiv 0 \bmod 2 \;:\; L(\tfrac{1}{2}, \xi_k) \neq 0\right\} \geq 75\%.$$

In the case $\alpha = 1$, we have

$$\lim_{K\to\infty} \frac{4}{K} \# \left\{1 \leq k \leq K,\, k \equiv 1 \bmod 4 \;:\; L(\tfrac{1}{2}, \xi_k) \neq 0\right\} \geq 25\%.$$

When $\alpha = 3$, each $L(s, \xi_k)$ vanishes at $s = \frac{1}{2}$, but we have

$$\lim_{K\to\infty} \frac{4}{K} \#\left\{1 \leq k \leq K,\, k \equiv 3 \bmod 4 \;:\; \operatorname{ord}_{s=\frac{1}{2}} L(s, \xi_k) = 1\right\} \geq 75\%.$$

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Agréez etc.