Bohr and Measure Recurrent Sets

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Return time sets

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implying

$$\mu(U \cap T^{-k}(U)) > 0$$

for some $k \in [m, m+n]$ and that $[m, m+n] \cap N(U, U) \neq \emptyset$.

Reminder: Return time sets and sets of positive density

Given a set $S \subset \mathbb{N}$, its upper density is given by

$$\overline{d}(S) := \limsup_{n \to \infty} \frac{S \cap [1, n]}{n}.$$

Given a set S of positive upper density, there is an

$$N(U, U) \subset S - S.$$

Further by the ergodic theorem we have that there is set of positive upper density $P \subset \mathbb{N}$ such that

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Thus studying return time sets is the same as studying difference sets of sets of positive upper density.

How did I get interested?

- Such sets were studied to give an ergodic theoretic proof of Szemeredi's theorem. [Furstenberg, 1976]
- If S is a return time set then given any zero-entropy process $X_i; i \in \mathbb{Z}$, X_0 can be predicted by $X_i; i \in S$. [C. , Weiss, 2019]

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Question (C. , Weiss, 2019)

Does every return time set S contain a return time set arising from a zero entropy process?

$$N(U,U) := \{n \in \mathbb{Z} \ : \ \mu(T^{-n}(U) \cap U) > 0\}.$$

What kind of sets are return-time sets?

Measure Recurrent Sets

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So far we have known that return time sets must have bounded gaps.

A good way of studying a set of a particular type is to study its *— A set $S \subset \mathbb{N}$ is called a measure recurrent set if it intersects every return time set, that is,

$$S \cap N(U, U) \neq \emptyset \text{ for all } N(U, U)'s.$$

These are also called Poincaré sets.

Even numbers, Return time sets are measure recurrent sets

$S \cap N(U, U) \neq \emptyset$ for all $N(U, U)'s$. -measure recurrent set

If (X, μ, T) is a ippt then so is (X, μ, T^2) is also a ippt. Thus there exists, by Poincaré recurrence theorem, some $2n \in 2\mathbb{N}$ such that $\mu(T^{-2n}(U) \cap U) > 0$. So the set of even integers is a measure recurrent set.

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More generally, given another ippt (Y, ν, S) and $V \subset Y$ of positive measure, we have that

$$N_T(U, U) \cap N_S(V, V) = N_{T \times S}(U \times V, U \times V)$$

for all $N_T(U, U)$. Thus all return times sets are measure recurrent sets.

Odd numbers are not measure recurrent sets

$S \cap N(U, U) \neq \emptyset$ for all $N(U, U)'s$. -measure recurrent set

However the odd numbers $2\mathbb{N} + 1$ are not measure recurrent set:

Let μ be the uniform probability measure on $\{0, 1\}$ and let $T : \{0, 1\} \to \{0, 1\}$ be given by $T(i) = i + 1 \mod 2$. Then for $U := \{0\}$, we have that $N(U, U) = 2\mathbb{Z}$.

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To see why this is true, let (X, μ, T) be a ppt and $U \subset X$ be of positive measure. Since $\mu(X) = 1$ and S is infinite there must exist distinct $s, s' \in S$ such that $\mu(T^s(U) \cap T^{s'}(U)) > 0$. Thus $|s - s'| \in N(U, U)$ and that $(S - S) \cap \mathbb{N}$ is a measure recurrent set.

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We will need the spectral representation: There is a finite non-negative measure ν_U on $\mathbb{R}/\mathbb{Z}$ such that $\nu_U(\{0\}) = \mu(U)^2$ and

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We will also need the following result of Weyl: For all irrational $t \in \mathbb{R}/\mathbb{Z}$

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Recall $\nu_U(\{0\}) = \nu_{rat}(\{0\}) = \mu(U)^2 > 0$. Since ϵ can be made arbitrarily small there exists $k \in \mathbb{N}$ such that

$$\mu(U \cap T^{-k^2m^2}(U)) > 0.$$

Intersective polynomials

Thus we have shown that the squares are measure recurrent sets (Sarkozy-Furstenberg theorem).

This proof was given by Furstenberg. He, in fact, showed that if p is a polynomial with rational coefficients such that $p(\mathbb{N}) \subset \mathbb{N}$ then $p(\mathbb{N})$ is a measure recurrent set if and only if p has a root modulo n for all n.

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Such polynomials are called intersective polynomials.

Return time sets: Summary

$$N(U,U) := \{n \in \mathbb{Z} \ : \ \mu(T^{-n}(U) \cap U) > 0\}$$ - return time sets

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But could they possibly be even more special?

Wiener's lemma

We will need the following consequence of Wiener's lemma.

Theorem

Let ν be a continuous non-negative finite measure on $\mathbb{R}/\mathbb{Z}$. Then for all $\epsilon > 0$,

$$\{n \in \mathbb{Z} \ : \ |\hat{\nu}(n)| > \epsilon\}$$

has zero density.

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There exists $a_i \geq 0$ and $t_i \in \mathbb{R}/\mathbb{Z}$ such that

$$\nu_U = \mu(U)^2\delta_0 + \sum_{k=1}^{\infty} a_k \delta_{t_k} + \nu_{con}$$

where ν_{con} is a continuous component of ν_U.

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$$P' := \left\{ n \in \mathbb{Z} \ : \ \left| \sum_{k=1}^{M} a_k \exp(-2\pi i n t_k) - \sum_{k=1}^{M} a_k \right| < \epsilon \right\}.$$

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Pick M large enough such that $\sum_{k>M} a_k < \epsilon$. Let

$$P' := \left\{ n \in \mathbb{Z} \;:\; |\sum_{k=1}^{M} a_k \exp(-2\pi i n t_k) - \sum_{k=1}^{M} a_k| < \epsilon \right\}.$$

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Return time sets and rotations of a torus

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We must have that

$$P \setminus Q \subset N(U, U).$$

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and

$$P \setminus Q \subset N(U, U).$$

By Wiener's lemma it follows that Q must have zero density.

Return time sets and rotations of a torus

We have that

$$P \setminus Q \subset N(U, U)$$

where Q has density zero and there is $\alpha \in (\mathbb{R}/\mathbb{Z})^M$ and

$$P = \{n \in \mathbb{Z} \ : \ |n\alpha| < \delta\}.$$

Such sets P are called Bohr neighbourhoods of 0.

We have proved a result which goes back to Bogolyubov (1939), Følner (1954) and Veech (1968):

Theorem

Every $N(U, U)$ set contains a Bohr neighbourhood of 0 barring a set of density zero.

Bohr topology

Let $\alpha \in (\mathbb{R}/\mathbb{Z})^M$ and $V \subset (\mathbb{R}/\mathbb{Z})^M$ be an open set. Let

$$N_\alpha(0, V) := \{n \in \mathbb{Z} \ : \ n\alpha \in V\}.$$

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Note that if $0 \in V$ and U is an open set such that $U - U \subset V$ then we have that for μ being the Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^M$

$$N_\alpha(U, U) := \{n \in Z \ : \ \mu((n\alpha + U) \cap U) > 0\} \subset N_\alpha(0, V).$$

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So, Bohr neighbourhoods of 0 are return time sets for rotations of the torus (in fact of any compact abelian group).

Bohr topology- Why do I care?

I care about return time sets and they are return times sets with a very concrete description.

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Theorem (Meyer, 1968)

If $P \subset \mathbb{N}$ is a Bohr closed set and μ is a measure on the circle such that $\hat{\mu}(n) = 0$ for $n \in \mathbb{N} \setminus P$ then μ is absolutely continuous.

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Such sets P have a deep relationship with prediction of zero entropy processes and go by the name Riesz sets. It is known that the squares are Bohr closed and hence are Riesz sets.

The cubes on the other hand are not a closed set and it is a wide open problem whether they are Riesz sets.

Return time sets and Bohr neighbourhoods

Theorem (Bogolyubov-Følner)

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Question

Does every return time set contain a Bohr neighbourhood of 0*?*

No!

Theorem (Kříž, 1968)

There exists a return time set which does not contain a Bohr neighbourhood of 0*.*

Return time sets and Bohr neighbourhoods

Even further is true.

Return time sets and Bohr neighbourhoods

Even further is true.

Theorem (Griesmer, 2020)

There exists a return time set which does not contain any Bohr open set.

Bohr recurrent sets

Recall that a set $S \subset \mathbb{N}$ is called measure recurrent if it intersects every return time set, that is, a set of the type $N(U, U)$.

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A set $S \subset \mathbb{N}$ is called Bohr recurrent set if it intersects every Bohr neighbourhood of zero, that is, a set of the type $N_\alpha(U, U)$.

If a set is measure recurrent then it is Bohr recurrent. By Kříž's theorem the converse is not true.

Thus we have that the following sets are measure recurrent and hence Bohr recurrent:

- For an infinite set S, the set $(S - S) \cap \mathbb{N}$.
- The squares, the cubes ....

Bohr recurrent sets

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The following was proved by Pollington (1979), de Mathan (1981) and Katznelson (1999).

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Lacunary sets are not Bohr recurrent set.

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They were all answering different question raised by Erdös. We will concentrate on Katznelson's version of the answer.

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For any growth rate slower than exponential there are examples of sets which are Bohr recurrent by Ajtai, Havas and Komlós (1983).

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Let $\rho > 1$. Fix a lacunary set $\{\lambda_i \ : \ i \in \mathbb{N}\}$ such that $\lambda_{i+1}/\lambda_i > \rho$. We want to find $\alpha \in (\mathbb{R}/\mathbb{Z})^d$ and $\epsilon > 0$ such that $|\lambda_i \alpha| > \epsilon$ for all $i \in \mathbb{N}$.

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Suppose $\rho > 5$. Let

$$\begin{aligned} A_i &:= \{\alpha \in \mathbb{R}/\mathbb{Z} \ : \ |\lambda_i \alpha| \geq 1/4\} \\ &:= \{k/\lambda_i \ : \ 1 \leq k \leq \lambda_i\} + \left[\frac{1}{4\lambda_i}, \frac{3}{4\lambda_i}\right] \end{aligned}$$

is a union of intervals of length $\frac{1}{2\lambda_i}$.

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is a union of intervals of length $\frac{1}{2\lambda_i}$. Since $\lambda_{i+1}/\lambda_i > 5$ we have that each such interval must contain at least two λ_{i+1} roots of unity. Thus $\cap A_i$ is non empty.

Lacunary sets are not Bohr recurrent

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We have shown that if $\rho > 5$ then there exists $\alpha \in \mathbb{R}/\mathbb{Z}$ such that $|\lambda_i \alpha| \geq 1/4$ for all i.

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For general $\rho > 1$, find d such that $\rho^d > 5$. Then $\lambda_{i+kd}; k \in \mathbb{N}$ is lacunary for all i and there exists $\alpha_i \in \mathbb{R}/\mathbb{Z}$ such that $|\lambda_{i+kd}\alpha_i| \geq 1/4$ for all $k \in \mathbb{N}$.

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$$|\lambda_i(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_d)| \geq 1/4$$

for all i. This completes the proof.

Katznelson proved more. He (and also Pollington and de Mathan) showed that there is dimension 1 set of $\alpha \in \mathbb{R}/\mathbb{Z}$ such that

$$\inf_i |\lambda_i \alpha| > 0.$$

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$$\inf_i |\lambda_i \alpha| > 0.$$

But this was not the end of his goal.

Recurrence and chromatic numbers

Divide $(\mathbb{R}/\mathbb{Z})^d$ into measurable parts each of radius less that $1/8$ labelled, say, $\{1, 2, \ldots, n\}$.

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Consider the graph structure on $\mathbb{Z}$ which is obtained by connecting m and n by an edge if and only if they differ by λ_i for some i.

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Thus we have a colouring of $\mathbb{Z}$ such that no two integers with the same colour are separated by a λ_i.

Consider the graph structure on $\mathbb{Z}$ which is obtained by connecting m and n by an edge if and only if they differ by λ_i for some i. Let us call the graph $\mathbb{Z}_\lambda$. Katznelson thus proved the following theorem:

Theorem

$\mathbb{Z}_\lambda$ has a finite chromatic number.

This was the question which Erdös had proposed.

Kříž's construction

Kříž's had shown that there are sets which are Bohr recurrent but not measure recurrent.

Kříž's construction

Kříž's had shown that there are sets which are Bohr recurrent but not measure recurrent. His construction involved giving a graph structure on $\mathbb{Z}$ in which

- Graphs with larger and larger chromatic numbers could be embedded.
- It had an independent set of positive density (subsets of $\mathbb{Z}$ where no two integers are joined by an edge).

Topological recurrence

A set $S \subset \mathbb{N}$ is called topologically recurrent if for all minimal systems (X, T) and open sets $U \subset X$ there exists $n \in S$ such that $U \cap T^{-n}(U)$ is non-empty.

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This connects to what we have discussed before by the following theorem:

Theorem

A set S is topologically recurrent if and only if the graph generated by it has an infinite chromatic number.

If a set is measure recurrent then it is topologically recurrent. If a set is topologically recurrent then it is Bohr recurrent. Lacunary sets are not topologically recurrent.

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Theorem

A set S is topologically recurrent if and only if the graph generated by it has an infinite chromatic number.

If a set is measure recurrent then it is topologically recurrent. If a set is topologically recurrent then it is Bohr recurrent. Lacunary sets are not topologically recurrent. By Kříž's construction there are sets which are Bohr recurrent but not measure recurrent.

The following question remains and goes back to Katznelson, Følner and Bogolyubov.

Question

Is there a set which is topologically recurrent but not Bohr recurrent?