Euler products inside the critical strip

Arshay Sheth

University of Warwick

CRG Seminar for L-functions
25th February, 2025

Theme of the talk

Let $L(s)$ be any L-function which

- has an Euler product for $\Re(s) > c$.
- admits analytic continuation
- has a functional equation relating values at $s \longleftrightarrow$ values at $k - s$.

The critical strip is the region $k - c < \Re(s) < c$, the critical line is the line $\Re(s) = \frac{k}{2}$ and the central point is the point $s = \frac{k}{2}$.

Key theme of the talk: There is strong sense in which Euler products should persist even inside the critical strip.

Outline of talk

1. Example: L-functions of elliptic curves
2. The general case: automorphic L-functions
3. Applications to Chebyshev's bias

Example: L-functions of elliptic curves

The original version of the Birch and Swinnerton-Dyer conjecture

Let $E/\mathbb{Q}$ be an elliptic curve with rank $\mathrm{rk}(E)$ and for each prime p, let $N_p = \#E_{\mathrm{ns}}(\mathbb{F}_p)$, where $E_{\mathrm{ns}}(\mathbb{F}_p)$ denotes the set of non-singular $\mathbb{F}_p$-rational points on a minimal Weierstrass model for E at p.

Conjecture (Birch and Swinnerton-Dyer)

We have that

$$\prod_{p \leq x} \frac{N_p}{p} \sim C(\log x)^{\mathrm{rk}(E)}$$

as $x \to \infty$ for some non-zero constant C depending only on E.

L-functions of elliptic curves

Let $a_p = p + 1 - N_p$ if $p \nmid N_E$ and $a_p = p - N_p$ if $p | N_E$. Then the L-function of E is defined for $\mathrm{Re}(s) > 3/2$ by

$$L(E,s) = \prod_{p|N_E} (1 - a_p p^{-s})^{-1} \cdot \prod_{p \nmid N_E} (1 - a_p p^{-s} + p^{1-2s})^{-1}.$$

- By the work of Wiles and Breuil–Conrad–Diamond–Taylor, $L(E,s)$ admits an analytic continuation to the entire complex plane and satisfies a functional equation which relates $L(E,s)$ to $L(E,2-s)$.
- In particular, the critical strip of $L(E,s)$ is the region $\frac{1}{2} < \Re(s) < \frac{3}{2}$ and the critical line of $L(E,s)$ is the line $\Re(s) = 1$.

Partial Euler product at $s = 1$

Let

$$P_E(x) = \prod_{\substack{p \le x \\ p \mid N_E}} \frac{1}{1 - a_p p^{-1}} \prod_{\substack{p \le x \\ p \nmid N_E}} \frac{1}{1 - a_p p^{-1} + p^{-1}}$$

to be the partial Euler product at $s = 1$. Then

$$P_E(x) = \prod_{p \le x} \frac{p}{N_p}$$

and so the conjecture can be reformulated to assert that

$$P_E(x) \sim \frac{1}{C(\log x)^{\mathrm{rk}(E)}} \text{ as } x \to \infty.$$

If $\mathrm{rk}(E) = 0$, the conjecture predicts that the Euler product should converge even at the central point.

Modern formulation of the Birch and Swinnerton–Dyer conjecture

Conjecture

Let $E/\mathbb{Q}$ *be an elliptic curve. Then*

$$\operatorname*{ord}_{s=1} L(E,s) = \operatorname{rk}(E).$$

Goldfeld's result

Theorem (Goldfeld)

Let $E/\mathbb{Q}$ be an elliptic curve. If

$$\prod_{p \leq x} \frac{N_p}{p} \sim C(\log x)^{\mathrm{rk}(E)}$$

as $x \to \infty$, then $L(E, s)$ satisfies the Riemann Hypothesis and $\operatorname*{ord}_{s=1} L(E, s) = \mathrm{rk}(E)$. Moreover, if we set $r := \operatorname*{ord}_{s=1} L(E, s)$, then

$$C = \frac{r!}{L^{(r)}(E, 1)} \cdot \sqrt{2} e^{r\gamma},$$

- Does the converse hold?
- Why is there a factor of $\sqrt{2}$? (if the partial Euler product at the centre converges to a non-zero value as $x \to \infty$, then its value is $L(E, 1)/\sqrt{2}$, as opposed to simply $L(E, 1)$).

Partial Euler products in the critical strip

What about the behaviour of partial Euler products in the right-half of the critical strip? (i.e. in the region $1 < \Re(s) < \frac{3}{2}$)?

Proposition (S.)

Assume the Riemann Hypothesis holds for $L(E,s)$. Then for a complex number $s \in \mathbb{C}$ with $1 < \Re(s) < \frac{3}{2}$, we have that

$$\prod_{\substack{p \leq x \\ p \mid N_E}} (1 - a_p p^{-s})^{-1} \cdot \prod_{\substack{p \leq x \\ p \nmid N_E}} (1 - a_p p^{-s} + p^{1-2s})^{-1} =$$

$$L(E,s) \cdot \exp\left(-r \cdot Li(x^{1-s}) - R_s(x) + U_s(x) + O\left(\frac{\log x}{x^{1/6}}\right)\right),$$

where $r = \operatorname*{ord}_{s=1} L(E,s)$, $\mathrm{Li}(x)$ *is the principal value of* $\int\limits_0^x \frac{dt}{\log t}$,

$R_s(x) = \frac{1}{\log x} \sum_{\rho \neq 1} \frac{x^{\rho - s}}{\rho - s} + \frac{1}{\log x} \sum_{\rho \neq 1} \int_s^\infty \frac{x^{\rho - z}}{(\rho - z)^2} dz$ *and* $U_s(x) = \sum_{\substack{\sqrt{x} < p \leq x \\ p \nmid N_E}} \frac{(\alpha_p^2 + \beta_p^2)}{2p^{2s}}$.

Analog of Ramanujan's result

Theorem (Ramanujan)

Assume the Riemann Hypothesis for $\zeta(s)$. For all $s \in \mathbb{C}$ with $\frac{1}{2} < \Re(s) < 1$ we have that

$$\prod_{p \le x} (1-p^{-s})^{-1} = -\zeta(s) \exp\left(\mathrm{Li}(\vartheta(x)^{1-s}) + \frac{2s x^{\frac{1}{2}-s}}{(2s-1)\log x} + \frac{S_s(x)}{\log x} + O\left(\frac{x^{\frac{1}{2}-s}}{\log(x)^2} \right) \right),$$

where $\mathrm{Li}(x)$ is the principal value of $\int_0^x \frac{dt}{\log t}$, $\vartheta(x) = \sum_{p \le x} \log p$ and

$S_s(x) = -s \sum_{\rho} \frac{x^{\rho-s}}{\rho(\rho-s)}$, the sum taken over all non-trivial zeros of $\zeta(s)$.

- Similar asymptotic by Kaneko (2022) for Dirichlet L-functions.
- Key technique in all the proofs: explicit formulas.

Logarithmic measure of sets

Definition

Let $S \subseteq \mathbb{R}_{\geq 2}$ be a measurable subset of the real numbers. The logarithmic measure of S is defined to be

$$\mu^{\times}(S) = \int_S \frac{dt}{t}.$$

Euler product asymptotics

Theorem (S.)

Assume the Riemann Hypothesis for $L(E,s)$. Then there exists a subset $S \subseteq \mathbb{R}_{\geq 2}$ of finite logarithmic measure such that for all $x \notin S$,

$$\prod_{p \leq x} \frac{N_p}{p} \sim C(\log x)^r,$$

where $r = \operatorname*{ord}_{s=1} L(E,s)$, $C = \frac{r!}{L^{(r)}(E,1)} \cdot \sqrt{2}e^{r\gamma}$, γ is Euler's constant and $L^{(r)}(E,s)$ is the r-th derivative of $L(E,s)$.

Proof Sketch: Set $s = 1 + \frac{1}{x}$ and let $x \to \infty$ in the proposition. The LHS is asymptotic to $\prod_{p \leq x} \frac{N_p}{p}$. The term $-r\mathrm{Li}(x^{1-s})$ contributes the main term $(\log x)^r$, and the term $U_s(x)$ contributes the factor of $\sqrt{2}$ appearing in the constant C. □

The contribution from the zeros

- The delicate issue in our proof is to handle the contribution coming from the zeros of $L(E,s)$ in the term $R_s(x)$.
- By another explicit formula argument, we reduce the problem to estimating the sum $\psi_E(x) = \sum_{\substack{p^k \leq x \\ p \nmid N_E}} (\alpha_p^k + \beta_p^k) \log p$, where α_p and β_p are the Frobenius eigenvalues at p.
- The Riemann Hypothesis for $L(E,s)$ is equivalent to $\psi_E(x) = O(x(\log x)^2)$.
- We use a method of Gallagher to obtain, conditional on the Riemann Hypothesis for $L(E,s)$, the slightly refined estimate $\psi_E(x) = O(x(\log\log x)^2)$ oustide a set of finite logarithmic measure.

Corollaries

As a corollary of the theorem, we recover Goldfeld's result as a special case:

Corollary

OBSD $\implies BSD.$

We also obtain a result in the direction of the converse

Corollary

BSD + RH for $L(E,s) \implies OBSD$ *outside a set of finite logarithmic measure.*

The general case: automorphic L-functions

Euler products in the critical strip

Theorem (Conrad)

If χ is a non-trivial Dirichlet character with associated Dirichlet L-function $L(s,\chi)$, then

$$\lim_{x\to\infty}\prod_{p\leq x}(1-\chi(p)p^{-s})^{-1}=L(s,\chi) \tag{0.1}$$

for all s with $Re(s)>\frac{1}{2}$ is equivalent to the Generalised Riemann Hypothesis for $L(s,\chi)$.

- Similar statement holds for entire L-functions.
- What about convergence on the critical line? It is believed that a similar statement should hold even on the critical line. However, at the central point, an unexpected factor of $\sqrt{2}$ if often known to appear.

Second moment L-functions

Let L-function $L(s)$, which we henceforth assume is normalised so its centre is at $s = \frac{1}{2}$, be given by an Euler product

$$L(s) = \prod_p \prod_{j=1}^{n} (1 - \alpha_{j,p} p^{-s})^{-1}.$$

Then its second moment L-function is given by

$$L_2(s) = \prod_p \prod_{j=1}^{n} (1 - \alpha_{j,p}^2 p^{-s})^{-1}$$

and in practice is the ratio of the corresponding symmetric square L-function and the exterior square L-function.

Examples of Conrad's theorem

Theorem (Conrad)

Let $R = \operatorname*{ord}_{s=1} L_2(s)$. *If the Euler product at the centre converges, then its value equals* $L(\frac{1}{2})/\sqrt{2}^R$.

Example

If χ is a Dirichlet character, $L_2(\chi, s) = L(\chi^2, s)$. Hence, if χ is a quadratic character, then $R = -1$; thus for a quadratic character, if $\lim_{x\to\infty} \prod_{p\leq x} (1 - \chi(p)p^{-1/2})^{-1}$ exists, then

$$\lim_{x\to\infty} \prod_{p\leq x} (1 - \chi(p)p^{-1/2})^{-1} = \sqrt{2} \cdot L\left(\frac{1}{2}, \chi\right).$$

Numerical evidence

Let χ_4 be the non-trivial character mod 4. Then $L\left(\chi_4, \frac{1}{2}\right) \approx 0.67$ and $\sqrt{2} \cdot L\left(\chi_4, \frac{1}{2}\right) \approx 0.94$.

x	$\prod_{p \leq x}(1 - \chi(p)p^{-1/2})^{-1}$
100	0.94
1000	0.89
10000	0.98
100000	0.97

Why does $\sqrt{2}$ appear?

Theorem (Conrad)

There is a constant M such that

$$\sum_{p \leq x} \frac{\alpha_{1,p}^2 + \cdots + \alpha_{n,p}^2}{p} = -R \log\log x + M + o(1).$$

Via explicit formula arguments, a term that comes up is

$$\sum_{p \leq x} \frac{\alpha_{1,p}^2 + \cdots + \alpha_{n,p}^2}{2p} - \sum_{p \leq \sqrt{x}} \frac{\alpha_{1,p}^2 + \cdots + \alpha_{n,p}^2}{2p}$$

This equals

$$\left(-\frac{R}{2} \log\log x + M + o(1)\right) - \left(-\frac{R}{2} \log\log \sqrt{x} + M + o(1)\right) = -R \log \sqrt{2} + o(1).$$

Kurokawa's conjecture

- Based on the above phenomena Kurokawa formulated a general conjecture about the convergence of partial Euler products at the centre of the critical strip.
- We now explain this conjecture in the setting of general automorphic L-functions attached to an irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$.
- Any automorphic L-function can be written in the form

$$L(s,\pi) = \prod_p \prod_{j=1}^{n} (1 - \alpha_{j,p} p^{-s})^{-1},$$

  where, for the unramified primes p, the $\alpha_{j,p}$'s are the Satake parameters for the local representation π_p.
- Ramanujan–Petersson conjecture: for any p such that π_p is uramified, $|\alpha_{j,p}| = 1$ for all $j \in \{1, \ldots, n\}$.

Kurokawa's conjecture

Conjecture (Kurokawa)

Keep the assumptions and notation as above. Let $m = \operatorname{ord}_{s=1/2} L(s, \pi)$*. The limit*

$$\lim_{x \to \infty} \left((\log x)^m \prod_{p \leq x} \prod_{j=1}^{n} \left(1 - \alpha_{j,p} p^{-\frac{1}{2}}\right)^{-1} \right)$$

satisfies the following conditions:

(A) The above limit exists and is non-zero.

(B) It satisfies

$$\lim_{x \to \infty} \left((\log x)^m \prod_{p \leq x} \prod_{j=1}^{n} \left(1 - \alpha_{j,p} p^{-\frac{1}{2}}\right)^{-1} \right) = \frac{\sqrt{2}^{-R(\pi)}}{e^{m\gamma} m!} L^{(m)} \left(\frac{1}{2}, \pi\right),$$

where $R(\pi) = \operatorname{ord}_{s=1} L_2(s, \pi)$*.*

Evidence for the conjecture

- Ample numerical evidence
- Function field analog is known.

Relation to error terms

Let $a_\pi(p^k) = \alpha_{1,p}^k + \cdots + \alpha_{n,p}^k$ and let

$$\psi(x,\pi) = \sum_{p^k \le x} \log p \cdot a_\pi(p^k).$$

Then the Generalised Riemann Hypothesis is equivalent to the estimate

$$\psi(x,\pi) = O(\sqrt{x}(\log x)^2)$$

while Kurokawa's Conjecture is equivalent to the estimate

$$\psi(x,\pi) = o(\sqrt{x}\log x).$$

In this quantitative sense, Kurokawa's conjecture seems deeper than the Generalised Riemann Hypothesis.

Relation to GRH

Error term may not be the best way to determine the precise relation between the Generalised Riemann Hypothesis and Kurokawa's conjecture; for instance, the Generalised Riemann Hypothesis is also equivalent to the slightly weaker error term $\psi(x, \pi) = O(x^{\frac{1}{2}+\epsilon})$ for any $\epsilon > 0$.

> Can the Generalised Riemann Hypothesis can be related to the Euler product at the centre as well?

Euler products at the central point

Theorem (S.)

Let π *be an irreducible cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ *such that* $L(s,\pi)$ *is entire and let* $m = \operatorname*{ord}_{s=\frac{1}{2}} L(s,\pi)$*. Assume the Ramanujan–Petersson Conjecture and the Generalised Riemann Hypothesis for* $L(s,\pi)$*. Then there exists a subset* $S \subseteq \mathbb{R}_{\geq 2}$ *of finite logarithmic measure such that for all* $x \notin S$*,*

$$(\log x)^m \cdot \prod_{p \leq x} \prod_{j=1}^{n} (1 - \alpha_{j,p} p^{-\frac{1}{2}})^{-1} \sim \frac{\sqrt{2}^{-R(\pi)}}{e^{m\gamma} m!} \cdot L^{(m)}\left(\frac{1}{2}, \pi\right).$$

Applications to Chebyshev's bias

Chebyshev's bias

Chebyshev's bias originally referred to the phenomenon that, even though the primes are equidistributed in the multiplicative residue classes mod 4, there seem to be more primes congruent to 3 mod 4 than 1 mod 4.

Let $\pi(x; q, a)$ denote the number of primes up to x congruent to a modulo q and let $S = \{x \in \mathbb{R}_{\geq 2} : \pi(x; 4, 3) - \pi(x; 4, 1) > 0\}$. Knapowski–Turán conjectured that the proportion of postive real numbers lying in the set S would equal 1 as $x \to \infty$. However, this conjecture was later disproven by Kaczorowski conditionally on the Generalised Riemann Hypothesis, by showing that the limit does not exist.

Rubinstein–Sarnak approach

Rubinstein and Sarnak instead considered the logarithmic density

$$\delta(S) := \lim_{X\to\infty} \frac{1}{\log X} \cdot \int_{t\in S\cap[2,X]} \frac{dt}{t}$$

.

Theorem (Rubinstein–Sarnak)

Assume the Generalised Riemann Hypothesis and that the non-negative imaginary parts of zeros of Dirichlet L-functions are linearly independent over $\mathbb{Q}$. Then

$$\delta(S) = 0.9959\ldots$$

Chebyshev's bias

Definition (Aoki–Koyama)

Let $(c_p)_p \subseteq \mathbb{R}$ *be a sequence over primes* p *such that*

$$\lim_{x\to\infty} \frac{\#\{p \mid c_p > 0, p \leq x\}}{\#\{p \mid c_p < 0, p \leq x\}} = 1.$$

We say that $(c_p)_p$ *has a Chebyshev bias towards being positive if there exists a positive constant* C *such that*

$$\sum_{p \leq x} \frac{c_p}{\sqrt{p}} \sim C \log\log x.$$

On the other hand, we say that c_p *is unbiased if*

$$\sum_{p \leq x} \frac{c_p}{\sqrt{p}} = O(1).$$

Chebyshev's bias for Satake parameters

Theorem (S.)

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ such that $L(s,\pi)$ is entire and let $m = \mathrm{ord}_{s=1/2} L(s,\pi)$. Assume the Ramanujan–Petersson Conjecture and the Riemann Hypothesis for $L(s,\pi)$. Then there exists a constant c_π such that

$$\sum_{p \le x} \frac{\alpha_{1,p} + \cdots + \alpha_{n,p}}{\sqrt{p}} = \left(\frac{R(\pi)}{2} - m \right) \log\log x + c_\pi + o(1)$$

for all x outside a set of finite logarithmic measure, where $R(\pi) = \operatorname*{ord}_{s=1} L_2(s,\pi)$.

Idea of proof: Take logarithm of the Euler product asymptotic at the central point.

Example # 1

Theorem (Aoki–Koyama)

Let χ_4 denote the non-trivial Dirichlet character modulo 4. Assume that Kurokawa's conjecture holds for $L(\chi_4, s)$. Then there exists a constant c such that

$$\sum_{p \leq x} \frac{\chi_4(p)}{\sqrt{p}} = -\frac{1}{2} \log \log x + c + o(1)$$

In particular, in the sense of the previous definition, there is a Chebyshev bias towards primes congruent to 3 mod 4.

Theorem (S.)

The same asymptotic holds outside a set of finite logarithmic measure assuming only GRH.

Example # 2

Theorem (Koyama–Kurokawa)

Let $\tau(n)$ denote Ramanujan's tau function. Assume Kurokawa's conjectire for $L(s,\Delta)$. Then there exists a constant c such that

$$\sum_{p \leq x} \frac{\tau(p)}{p^6} = \frac{1}{2} \log\log x + c + o(1).$$

In particular, the sequence $\tau(p)p^{-\frac{11}{2}}$ has a Chebyshev bias towards being positive.

Theorem (S.)

The same asymptotic holds outside a set of finite logarithmic measure assuming only GRH.

Work in progress with Koyama: work out the case for general modular forms.

Thank you!