Prime Number Error Terms

Nathan Ng

University of Lethbridge

Comparative Prime Number Theory Symposium
CRG: L-functions in Analytic Number Theory
University of British Columbia
June 17, 2024

Summatory functions

$$\psi(x) = \sum_{n \le x} \Lambda(n), \quad M(x) = \sum_{n \le x} \mu(n), \quad L(x) = \sum_{n \le x} \lambda(n).$$

- Von Koch (1905) RH implies

$$\frac{\psi(x) - x}{\sqrt{x}} = O(\log^2 x).$$

- Littlewood (1914)

$$\frac{\psi(x) - x}{\sqrt{x}} = \Omega_{\pm}(\log \log \log x).$$

- Bui-Florea (2023) RH implies

$$\frac{M(x)}{\sqrt{x}} = O\left(\exp(\sqrt{\log x}(\log \log x)^{\frac{7}{8}+\varepsilon} \right)$$

- Hurst (2018)

$$\limsup_{x \to \infty} \frac{M(x)}{\sqrt{x}} > 1.826054 \text{ and } \liminf_{x \to \infty} \frac{M(x)}{\sqrt{x}} < -1.837625.$$

Error term Conjectures

Conjecture (Montgomery, 1980)

$$\overline{\lim}_{x\to\infty} \frac{\psi(x) - x}{\sqrt{x}(\log\log\log x)^2} = \pm\frac{1}{2\pi} \ .$$

Conjecture (Ng, 2012)

$$\overline{\lim}_{x\to\infty} \frac{M(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} = \pm\frac{8a}{5} \ . \qquad (1)$$

- $a = \frac{1}{\sqrt{\pi}} e^{3\zeta'(-1) - \frac{11}{12}\log 2} \prod_p \left((1 - p^{-1})^{\frac{1}{4}} \sum_{k=0}^{\infty} \left(\frac{\Gamma(k - \frac{1}{2})}{k!\Gamma(-\frac{1}{2})} \right)^2 p^{-k} \right)$.
- $a = 0.16712\ldots$ arises from a conjecture of Hughes, Keating, O'Connell.
- Gonek (1990's) conjectured (1) with an unspecified constant.

Linear Independence

Conjecture (LI: Linear Independence, Wintner 1935)

The positive ordinates of the zeros of the Riemann zeta function are linearly independent over the rational numbers.

Conjecture (ELI: Effective Linear Independence)

Let $\{\gamma\}$ *denote the set of the ordinates of the non-trivial zeros of the Riemann zeta function. For every* $\varepsilon > 0$ *there exists a positive constant* C_ε *such that for all real numbers* $T \geq 2$ *we have*

$$\Big| \sum_{0<\gamma\leq T} \ell_\gamma \gamma \Big| \geq C_\varepsilon e^{-T^{1+\varepsilon}},$$

where the ℓ_γ *are integers, not all zero, such that* $|\ell_\gamma| \leq N(T)$*, and* $N(T)$ *is the number of non-trivial zeros of* $\zeta(s)$ *with imaginary part in* $(0, T]$.

- Conjectured by Monach-Montgomery (see Montgomery-Vaughan p. 483)
- Heuristic arguments: Damien Roy (2018), Lamzouri (2023).

Theorem (Lamzouri, 2023)

Assume ELI. Then we have

$$\limsup_{x\to\infty} \frac{\psi(x)-x}{\sqrt{x}(\log\log\log x)^2} \geq \frac{1}{2\pi} \text{ and } \liminf_{x\to\infty} \frac{\psi(x)-x}{\sqrt{x}(\log\log\log x)^2} \leq -\frac{1}{2\pi}.$$

Theorem (Lamzouri, 2023)

Assume ELI,

$$\sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|} \ll T(\log T)^{1/4}, \quad \text{and} \sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{1.267}.$$

Then we have

$$M(x) = \Omega_{\pm}\left(\sqrt{x}(\log\log\log x)^{5/4}\right).$$

Explicit formulae and random sums

- If $\sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{1.999}$, then

$$\frac{M(x)}{\sqrt{x}} = 2\mathfrak{Re}\left(\sum_{0<\gamma\leq X^2} \frac{x^{i\gamma}}{\rho\zeta'(\rho)}\right) + O(1) \text{ for } 2 \leq x \leq X.$$

- Variable change $x = e^t$

$$\frac{M(e^t)}{\sqrt{e^t}} \sim 2\mathfrak{Re}\left(\sum_{0<\gamma\leq X^2} \frac{e^{ti\gamma}}{\rho\zeta'(\rho)}\right) = 2\mathfrak{Re}\left(\sum_{0<\gamma\leq X^2} \frac{e^{ti\gamma+i\beta_\gamma}}{|\rho\zeta'(\rho)|}\right).$$

- If LI is true, Kronecker-Weyl theorem suggests sum behaves like the random sum

$$\mathbf{X}(\underline{\theta}) = 2\mathfrak{Re}\left(\sum_{0<\gamma\leq X^2} \frac{e^{2\pi i\theta_\gamma}}{|\rho\zeta'(\rho)|}\right)$$

where $\underline{\theta} = (\theta_{\gamma_1}, \theta_{\gamma_2}, \ldots) \in \mathbb{T}^N$ and $N \in \mathbb{N}$.

Explicit formula and general sums over zeros

Let

$$\Phi_{X,\mathbf{r}}(x) := \mathfrak{Re}\left(\sum_{0<\gamma\leq X} x^{i\gamma} r_\gamma\right)$$

where $\mathbf{r} = \{r_\gamma\}_{\gamma>0}$ is a complex sequence satisfying:

A1: There exist $\alpha_+, \alpha_-, A > 0$ such that

$$\alpha_-(\log T)^A \leq \sum_{0<\gamma\leq T} |r_\gamma| \leq \alpha_+(\log T)^A \text{ as } T \to \infty.$$

A2:

$$\sum_{0<\gamma\leq T} \gamma|r_\gamma| = o\big(T(\log T)^A\big)$$

A3:

$$\sum_{0<\gamma\leq T} \gamma^2|r_\gamma|^2 \ll T^\theta \text{ where } \theta < 1.999.$$

- $\frac{\psi(x)-x}{\sqrt{x}} \approx \Phi_{X,\mathbf{r}}(x)$ when $r_\gamma = \frac{1}{\rho}$.
- $\frac{M(x)}{\sqrt{x}} \approx \Phi_{X,\mathbf{r}}(x)$ when $r_\gamma = \frac{1}{\rho\zeta'(\rho)}$.
- $\frac{L(x)}{\sqrt{x}} \approx \Phi_{X,\mathbf{r}}(x)$ when $r_\gamma = \frac{\zeta(2\rho)}{\rho\zeta'(\rho)}$.

Theorem (Lamzouri, 2023)

Assume ELI. Let $\{r_\gamma\}_{\gamma>0}$ *be a sequence of complex numbers satisfying* **A1**, **A2**, **A3***'. Let* X *be large. There exist positive constants* C_1 *and* C_2 *such that*

$$\max_{x\in[2,X]} \Phi_{X^2,\mathbf{r}}(x) \ge C_1(\log\log\log X)^A.$$

and

$$\min_{x\in[2,X]} \Phi_{X^2,\mathbf{r}}(x) \le -C_2(\log\log\log X)^A.$$

- **A1**: $\sum_{0<\gamma\le T} |r_\gamma| \asymp (\log T)^A$
- **A2**: $\sum_{0<\gamma\le T} \gamma|r_\gamma| = o\big(T(\log T)^A\big)$
- **A3**': $\sum_{0<\gamma\le T} \gamma^2|r_\gamma|^2 \ll T^\theta$ where $\theta < 1.267$

General Theorem 2: $\Phi_{X,\mathbf{r}}(x) := \Re\mathfrak{e}\left(\sum_{0<\gamma\leq X} x^{i\gamma} r_\gamma\right)$

Theorem (Ng, 2024)

Assume ELI. Let $\{r_\gamma\}_{\gamma>0}$ be a sequence of complex numbers satisfying **A1**, **A2**, **A3**. *Let X be large. Then*

$$\max_{x\in[2,X]} \Phi_{X^2,\mathbf{r}}(x) \geq \alpha_-(\log\log\log X)^A$$

and

$$\min_{x\in[2,X]} \Phi_{X^2}(x) \leq -\alpha_-(\log\log\log X)^A.$$

- **A1**: $\alpha_-(\log T)^A \leq \sum_{0<\gamma\leq T} |r_\gamma| \leq \alpha_+(\log T)^A$
- **A2**: $\sum_{0<\gamma\leq T} \gamma|r_\gamma| = o\big(T(\log T)^A\big)$
- **A3**: $\sum_{0<\gamma\leq T} \gamma^2|r_\gamma|^2 \ll T^\theta$ *where* $\theta < 2$

Remarks: (i) Minor modifications of Lamzouri's result. Better lower bound for $I_0(t)$ and adjustment of various parameters in proof.
(ii) Lamzouri had unspecified constants C_1, C_2 instead of $\pm\alpha_-$.
(iii) Weakened condition in **A3**' from $\theta < 1.267$ to $\theta < 2$ using idea of Meng.

Application to $M(x)$

Theorem (Ng, 2024)

Assume ELI,

$$\sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|} \sim aT(\log T)^{1/4}, \quad \textit{and} \quad \sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{1.999}.$$

Then we have

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} \geq \frac{8a}{5} \quad \textit{and} \quad \liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} \leq -\frac{8a}{5}$$

- Apply previous theorem with $r_\gamma = \frac{1}{\rho\zeta'(\rho)}$.
- $\frac{8}{5} = \frac{4}{5} \cdot 2$ $\quad$ $\frac{4}{5}$ partial summation, 2 zero symmetry.
- Weak Mertens Conjecture $\implies \sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|^2} = o(T^2)$
- Bui-Florea-Milinovich. RH $\implies \sum_{\substack{0<\gamma<T \\ \gamma\in S}} \frac{1}{|\zeta'(\rho)|^2} = O(T^{1.51})$ for certain S.

Application to $L(x) = \sum_{n \le x} \lambda(n)$

Theorem (Ng, 2024)

Assume ELI,

$$\sum_{0<\gamma<T} \frac{|\zeta(2\rho)|}{|\zeta'(\rho)|} \sim bT(\log T)^{1/4}, \quad \text{and} \quad \sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{1.999}.$$

Then we have

$$\limsup_{x\to\infty} \frac{L(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} \ge \frac{8b}{5} \text{ and } \liminf_{x\to\infty} \frac{L(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} \le -\frac{8b}{5}$$

- Apply previous theorem with $r_\gamma = \frac{\zeta(2\rho)}{\rho\zeta'(\rho)}$.

Conjecture (Akbary, Ng, Yang Li, 2012)

$$b = a \cdot \sum_{n=1}^{\infty} \frac{d_{\frac{1}{2}}(n)^2}{n^2}$$

Discrete moment conjectures

Conjecture (Hughes, Keating, O'Connell, 2000)

For $0 \leq s < 3$,

$$\sum_{0<\gamma_n<T} \frac{1}{|\zeta'(\rho)|^s} \sim \frac{G^2(2-\frac{s}{2})}{G(3-s)} a(-\tfrac{s}{2}) \frac{T}{2\pi}\left(\log \frac{T}{2\pi}\right)^{(\frac{s}{2}-1)^2}$$

where G *is Barnes' function,* $a(x) = \prod_p (1-\frac{1}{p})^{x^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+x)}{m!\Gamma(x)}\right)^2 p^{-m}$.

Conjecture (Akbary, Ng, Yang Li, 2012)

For $0 \leq s < 3$,

$$\sum_{0\leq\gamma\leq T} \left|\frac{\zeta(2\rho)}{\zeta'(\rho)}\right|^s \sim \frac{G^2(2-\frac{s}{2})}{G(3-s)} a(-\tfrac{s}{2}) \left(\sum_{n=1}^{\infty} \frac{d_{s/2}(n)}{n^2}\right) \frac{T}{2\pi}\left(\log \frac{T}{2\pi}\right)^{(\frac{s}{2}-1)^2}$$

where $d_k(\cdot)$ *is the* k*-th divisor function.*

$$\sum_{0\leq\gamma\leq T} \left|\frac{\zeta(2\rho)}{\zeta'(\rho)}\right| \sim bT(\log T)^{\frac{1}{4}} \text{ and } \sum_{0\leq\gamma_n\leq T} \left|\frac{\zeta(2\rho)}{\zeta'(\rho)}\right|^2 \sim \frac{T}{2\pi}.$$

A general conjecture

Let φ be real-valued with an "explicit formula"

$$\varphi(t) = c_0 + 2\mathfrak{Re} \sum_{0<\gamma_n<T} r_{\gamma_n} e^{i\gamma_n t} + \mathcal{E}(t,T)$$

where $c_0 \in \mathbb{R}$, $\mathcal{E}(t,T)$ satisfies a certain mean square bound, and

$$\sum_{0<\gamma_n\leq T} 2|r_{\gamma_n}| \sim \alpha(\log T)^{\beta}.$$

Conjecture (Akbary, Ng, Shahabi, 2012)

$$\limsup_{x\to\infty} \frac{\varphi(\log x)}{(\log\log\log x)^{\beta}} = \alpha \text{ and } \liminf_{x\to\infty} \frac{\varphi(\log x)}{(\log\log\log x)^{\beta}} = -\alpha.$$

$\varphi(t)$	r_γ	$\sum_{0<\gamma<T} 2r_\gamma$	α	β
$\frac{\psi(e^t)-e^t}{e^{t/2}}$	$\frac{1}{\rho}$	$\frac{1}{2\pi}(\log T)^2$	$\frac{1}{2\pi}$	2
$\frac{M(e^t)}{e^{t/2}}$	$\frac{1}{\rho\zeta'(\rho)}$	$\frac{8a}{5}(\log T)^{\frac{5}{4}}$	$\frac{8a}{5}$	$\frac{5}{4}$
$\frac{L(e^t)}{e^{t/2}}$	$\frac{\zeta(2\rho)}{\rho\zeta'(\rho)}$	$\frac{8b}{5}(\log T)^{\frac{5}{4}}$	$\frac{8b}{5}$	$\frac{5}{4}$

Random sums and large deviations

Let $\mathbf{r} = \{r_{\gamma_n}\}_{n=1}^{\infty}$ and consider the associated random sum

$$\mathbf{X}_{\mathbf{r}} = 2\sum_{n=1}^{\infty} r_{\gamma_n} \cos(2\pi\theta_n)$$

where $\theta_n \in [0,1]$ are IID random variables. Assume $\sum_{0<\gamma_n\le T} 2|r_{\gamma_n}| \sim \alpha(\log T)^{\beta}$ and 4 other conditions on r_{γ_n}, γ_n.
(r_{γ_n} NOT NECESSARILY DECREASING.)

Theorem (Akbary, Ng, Majid Shahabi, 2012, unpublished)

Let $\varepsilon > 0$. Then for $V \ge V_\varepsilon$, we have

$$\exp\left(-\exp\left(\left(\alpha^{\frac{1}{\beta}}+\varepsilon\right)V^{\frac{1}{\beta}}\right)\right) \le P\left(\mathbf{X}_{\mathbf{r}} \ge V\right) \le \exp\left(-\exp\left(\left(\alpha^{\frac{1}{\beta}}-\varepsilon\right)V^{\frac{1}{\beta}}\right)\right).$$

- Upper bound: Montgomery, Odlyzko (Acta. Arith., 1988), Theorem 2 (Chernoff's inequality).
- Lower bound: Montgomery (Queen's Conf., 1980), Sec. 3, Theorem 1.
- Shahabi's M.Sc. thesis has more precise results for $P\left(\mathbf{X}_{\mathbf{r}} \ge V\right)$ similar to Granville-Lamzouri (2021).
- Theorem shows why we don't expect to improve the lower bounds in omega theorems.

Lamzouri's argument

1. Let $F(t,T) = \sum_{0<\gamma\le T} \cos(\gamma t + \beta_\gamma)|r_\gamma|$.
2. For X large show there exists $t \in [1, X]$.

$$F(t, e^{2X}) = \sum_{0<\gamma\le e^{2X}} \cos(\gamma t + \beta_\gamma)|r_\gamma| \ge (\alpha_- - \varepsilon)(\log\log X)^2.$$

   Variable change $e^X \to X$, $t = \log x$ establishes Theorem.
3. ELI implies

$$\frac{1}{X}\int_1^X \exp(sF(t,T))\,dt \sim \mathbb{E}\Big(\exp\Big(s\sum_{0<\gamma\le T}|r_\gamma|\cos(\theta_\gamma)\Big)\Big) \qquad (2)$$

   for $T = (\log X)^{1-\varepsilon}$ (random moment generating function).
4. Probability ideas show RHS is large. Independence, bounds for I_0 Bessel functions, and insert lower bound for $\sum_{0<\gamma\le T}|r_\gamma|$.
5. Deduce from (2) that

$$F(t,T) \ge (\alpha_- - \varepsilon')(\log T)^A$$

   for many values of $t \in [1, X]$ for $T = (\log X)^{1-\varepsilon}$.

Lamzouri's argument cont'd

6. Use a smoothing to relate $F(t, (\log X)^{1-\varepsilon})$ to an average of $F(t+u, X)$ where $|u| \leq (\log X)^A$.
7. Show that $F(t+u, e^{2X}) - F(t+u, X)$ is small on average for $t \in [1, X]$. This allows one to obtain large values of $F(t+u, e^{2X})$ as desired. Requires following lemma.

Lemma (Ng, 2024)

Let $\{r_\gamma\}_{\gamma>0}$ be a sequence of complex numbers satisfying **A3***:*

$$\sum_{0<\gamma\leq T} \gamma^2 |r_\gamma|^2 \ll T^\theta \text{ where } \theta < 2.$$

There exists a positive constant $\alpha = \alpha(\theta)$, such that for all $X_2 > X_1 > 1$ we have

$$\sum_{X_1<\gamma_1,\gamma_2\leq X_2} |r_{\gamma_1} r_{\gamma_2}| \min\left(1, \frac{1}{|\gamma_1 - \gamma_2|}\right) \ll X_1^{-\alpha}.$$

- Variant of a lemma of Akbary, Ng, Shahabi (2014) using an idea of Meng (2017).

References

- A. Akbary, N. Ng, and M. Shahabi, *Limiting distributions of the classical error terms of prime number theory*, QJM, 2014.
- A. Akbary, N. Ng, and M. Shahabi, *Error terms in prime number theory and large deviations of sums of independent random variables*, preprint.
- A. Granville, Y. Lamzouri, *Large deviations of sums of random variables*. Lith. Math. J., 2021.
- C. P. Hughes, J. P. Keating and N. O'Connell, *Random matrix theory and the derivative of the Riemann zeta function*. Proc. Roy. Soc. London Ser. A, 2000
- Y. Lamzouri, *An effective Linear Independence conjecture for the zeros of the Riemann zeta function and applications*, `https://arxiv.org/abs/2311.04860`, 2023.
- X. Meng, *The distribution of k-free numbers and the derivative of the Riemann zeta-function*, MPCPS, 2017.
- W. R. Monach, *Numerical investigation of several problems in number theory*, Ph. D. Thesis, University of Michigan, 1980.
- H. L. Montgomery, *The zeta function and prime numbers*, Proceedings of the Queen's Number Theory Conference, 1979.
- H. L. Montgomery and A. M. Odlyzko, *Large deviations of sums of independent random variables*, Acta Arith., 1988.
- N. Ng, *The summatory function of the Möbius function*, PLMS, 2004.
- Majid Shahabi, *The distribution of the classical error terms in prime number theory*, Master's thesis, University of Lethbridge, 2012.