# On the generalised Dirichlet divisor problem

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## Definitions

Let $d_k(n)$ be the generalised divisor function.

$$\sum_{n \le x} d_k(n) = xP_{k-1}(\log x) + \Delta_k(x)$$

where $P_{k-1}(t)$ is a degree $k-1$ polynomial, and $\Delta_k(x)$ is a remainder term.

# Conjecture

**Conjecture**

For every $k \geq 2$, $\Delta_k(x) \ll_\varepsilon x^{1/2-1/(2k)+\varepsilon}$ holds for every $\varepsilon > 0$.

- Unproved for any $k \geq 2$
- This implies the Lindelöf Hypothesis

# Case $k$ large

## Karatsuba constant

When $k$ is large, the current best known bounds take the form

$$\Delta_k(x) \ll_\varepsilon x^{1-Dk^{-2/3}+\varepsilon},$$

where $D > 0$ is the Karatsuba constant.

- Under Richert's bound of the form $|\zeta(\sigma+it)| \ll t^{B(1-\sigma)^{3/2}} \log^{2/3} t$ uniformly for $1/2 \le \sigma \le 1$ and $B > 0$, there exists $c_0 > 0$ for which

$$\Delta_k(x) \ll_\varepsilon x^{1-Dk^{-2/3}+\varepsilon}, \quad D = c_0 B^{-2/3}$$

- $B = 4.43795$ (B., 2024)

# Literature review

$$\Delta_k(x) \ll_\varepsilon x^{1-Dk^{-2/3}+\varepsilon}$$

| Reference | $D$ | $k$ |
|---|---|---|
| Karatsuba (1972) | 0.117 | $k \geq 2$ |
| Ivić and Ouellet (1989) | 0.196 | $k > 10$ |
| $*$ Kolpakova (2011) | 0.283 | $k \geq 186$ |
| Heath-Brown (2017) | 0.849 | $k \geq 2$ |

- Instead of Richert's bound, Heath-Brown assumes

$$\zeta(\sigma + it) \ll_\varepsilon t^{B(1-\sigma)^{3/2}+\varepsilon}, \qquad 1/2 \leq \sigma \leq 1$$

with $B = 8\sqrt{15}/63 = 0.4918\ldots$.

# Statement of the new results

**Theorem 1 (B.-Yang, 2024)**

Let $k$ be a fixed positive integer. Then, for $k \geq 58$

$$\Delta_k(x) \ll x^{1-1.224(k-2.36)^{-2/3}}.$$

**Theorem 2 (B.-Yang, 2024)**

For all sufficiently large fixed $k$

$$\Delta_k(x) \ll x^{1-1.889k^{-2/3}}.$$

# Some preliminary tools

## Carlson's abscissa

For $k > 0$, Carlson's abscissa $\sigma_k$ is the infimum of numbers $\sigma$ for which for any $\varepsilon > 0$

$$\int_1^T |\zeta(\sigma + it)|^{2k} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}.$$

## Carlson's exponent

Carlson's exponent $m(\sigma)$ is the supremum of all $m \geq 4$ such that for any $\varepsilon > 0$

$$\int_1^T |\zeta(\sigma + it)|^{m} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}.$$

# Outline of the proof of Theorem 1

**Theorem 1 (B.-Yang, 2024)**

Let $k$ be a fixed positive integer. Then, for $k \geq 58$

$$\Delta_k(x) \ll x^{1-1.224(k-2.36)^{-2/3}}.$$

# Outline of the proof of Theorem 1

$$\boxed{\zeta(\sigma+it) \ll_\varepsilon t^{B(1-\sigma)^{3/2}+\varepsilon}}$$

Upper bound for Carlson's abscissa $\sigma_k$

$$\Downarrow$$

Lower bound for Carlson's exponent $m(\sigma)$

$$\Downarrow$$

Perron's formula on $\sum_{n\le x} d_k(n)$ $\quad + \quad$ Residue Theorem

# Outline of the proof of Theorem 1

Main idea of the proof is to find $\sigma_k$ such that for all $\sigma \geq \sigma_k$,

$$\int_1^T |\zeta(\sigma + it)|^{2k} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}.$$

Use an iterative method.

# Outline of the proof of Theorem 1

Find the smallest upper bound for $\sigma_k$ such that

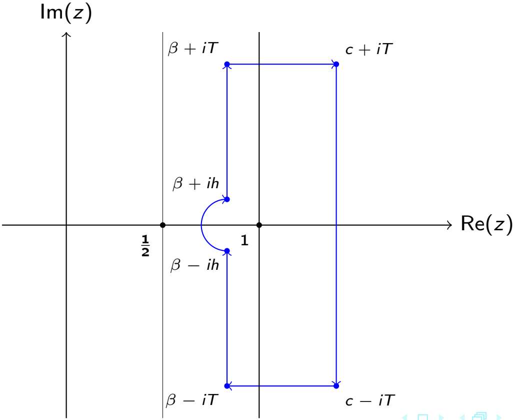
$$\int_1^T |\zeta(\sigma + it)|^{2k} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}, \qquad \sigma \geq \sigma_k.$$

Iterative method:

1. We wish to prove an upper bound on $\sigma_k$
2. Start with a bound on $\sigma_r$, for some $r < k$
3. Show that the bound on $\sigma_r$ implies a similar bound for $\sigma_{r+\delta}$ for some fixed $\delta > 0$

# Conclusion of the proof of Theorem 1

$$\Delta_k(x) \ll_\varepsilon T^\varepsilon \left( x^\beta T^{B(k-m_0(\beta))(1-\beta)^{3/2}} + x^\beta + \frac{x}{T} \right), \qquad T = x^{f(\beta)}.$$



# Outline of the proof of Theorem 2

**Theorem 2 (B.-Yang, 2024)**

For all sufficiently large fixed $k$

$$\Delta_k(x) \ll x^{1-1.889k^{-2/3}}.$$

# Outline of the proof of Theorem 2

Upper bound for Carlson's abscissa $\sigma_k$

$$\Downarrow$$

Lower bound for Carlson's exponent $m(\sigma)$

$$\Downarrow$$

Perron's formula on $\sum_{n\leq x} d_k(n)$ $+$ Residue Theorem

# Outline of the proof of Theorem 2

Main idea of the proof is to find $\sigma_k$ such that for all $\sigma \geq \sigma_k$,

$$\int_1^T |\zeta(\sigma + it)|^{2k} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}.$$

Use exponential sum estimates.

# Outline of the proof of Theorem 2

Main innovative idea of the proof is to use the approximate functional equation

$$\zeta(s) = \sum_{1 \le n \le T^{1/2}} n^{-s} + \chi(1-s) \sum_{1 \le n \le T^{1/2}} n^{1-s} + o(1)$$

and estimate

$$\int_T^{2T} \left| \sum_{n \le T^{1/2}} n^{-\sigma - it} \right|^{2k} \mathrm{d}t$$

using the mean value theorem for Dirichlet polynomials and exponential sum estimates.

## Outline of the proof of Theorem 2

1. Use Minkowski's inequality.
2. By mean value theorem $\int_T^{2T} \left| \sum_{n \leq T^{1/k}} n^{-\sigma - it} \right|^{2k} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}$.
3. For the second term, it suffices to prove that

$$\int_T^{2T} \left| \sum_{N \leq n \leq 2N} n^{-\sigma - it} \right|^{2k} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}, \qquad T^{1/k} < N \leq T^{1/2}.$$

## An exponential sum estimate

By refining an estimate due to Heath-Brown (2017),

$$\sum_{N<n\leq N'} n^{-it} \ll_{\varepsilon} N^{1-(1-3\rho^{-1})\rho^{-2}+\varepsilon}, \qquad \rho = \frac{\log N}{\log t} \geq 3$$

for $N < N' \leq 2N$.
Replaces the well-known result with $c = 49/80$ with $c = 1 - 3/\rho$.

For more details:

- C.Bellotti, A. Yang, *"On the generalised Dirichlet divisor problem"*, Bull. Lond. Math. Soc., 56.5 (2024)

Thank you for your attention!