Shifting the ordinates of zeros of the Riemann zeta function

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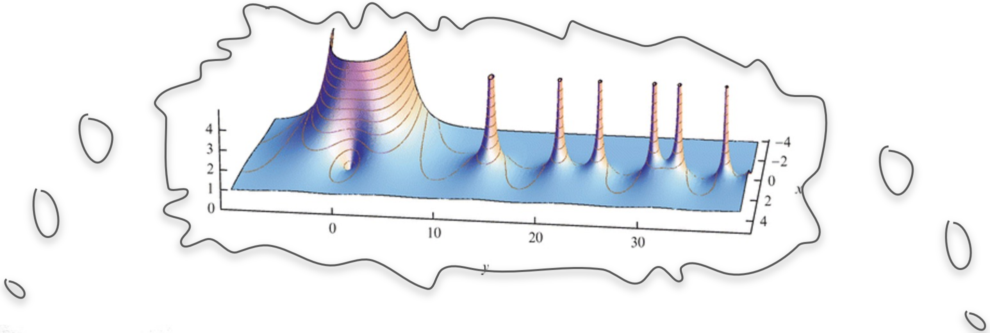
Comparative Prime Number Theory Symposium



Riemann zeta function

$$\zeta(s) := \sum_{n\geqslant 1} n^{-s} = \prod_{p \text{ prime}} (1-p^{-s})^{-1}$$

Dirichlet L-function

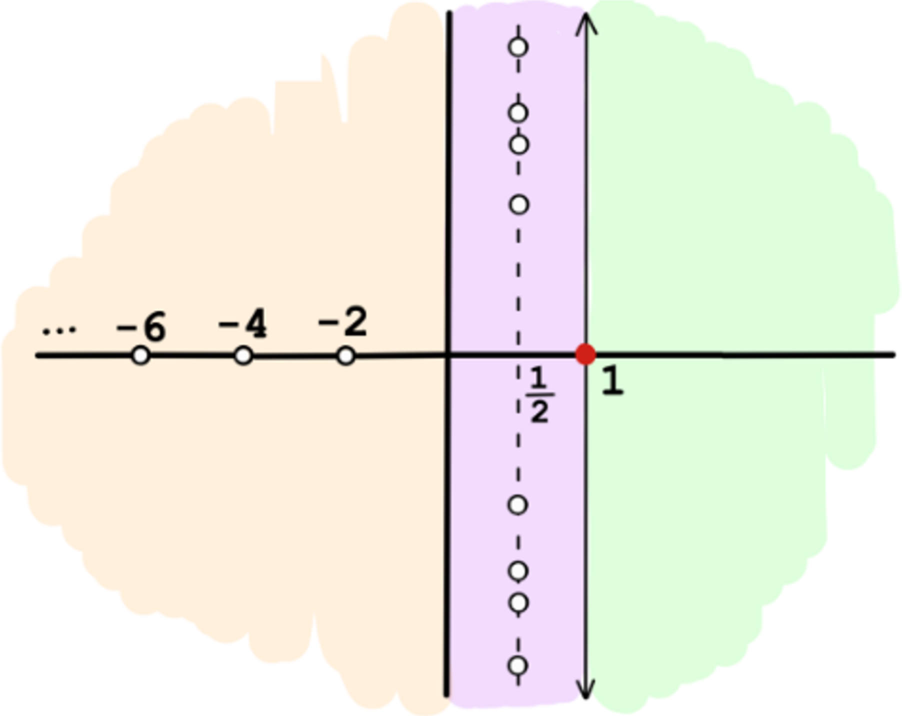
$$L(s,\chi) := \sum_{n\geqslant 1} \chi(n) n^{-s} = \prod_{p \text{ prime}} (1-\chi(p)p^{-s})^{-1}$$

Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

Riemann Hypothesis

$$\zeta(\sigma + it) \neq 0 \quad \text{if} \quad \sigma > \tfrac{1}{2}$$



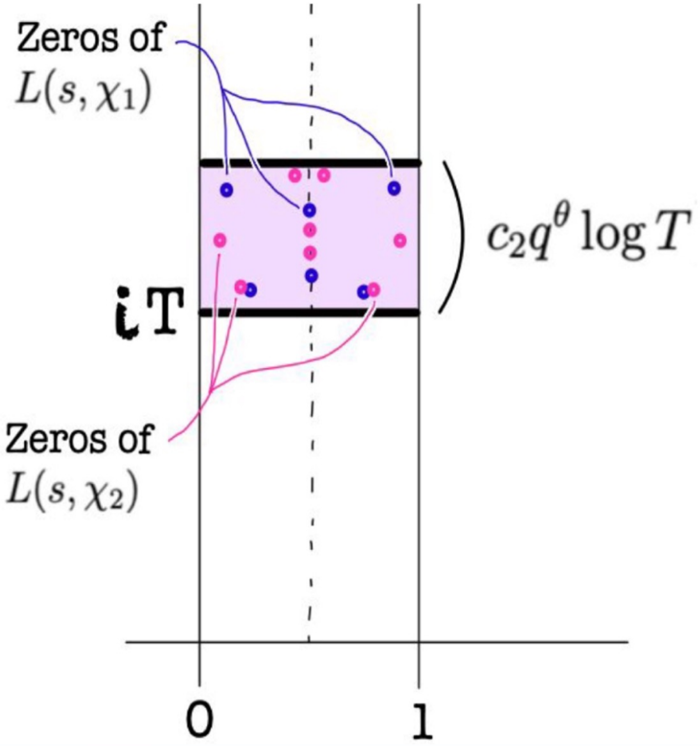
Theorem (B- 2023). *For any $\theta > \frac{1}{3}$, there are constants $c_1, c_2 > 0$ that depend only on θ such that the following property holds. Let χ_1 and χ_2 be distinct primitive Dirichlet characters modulo q. Then, for any $T \geqslant c_1 q^\theta$, the functions $L(s, \chi_1)$ and $L(s, \chi_2)$ have different zeros (counted with multiplicity) in the region*

$$\{s = \sigma + it \in \mathbb{C} : 0 < \sigma < 1,\ T < \gamma < T + c_2 q^\theta \log T\}.$$

In other words, the function

$$\frac{L(s,\chi_1)}{L(s,\chi_2)} + \frac{L(s,\chi_2)}{L(s,\chi_1)}$$

has at least one pole in the region.



COMPARING ZEROS OF DISTINCT DIRICHLET L-FUNCTIONS

Landau (1912)

$$\sum_{\substack{\rho=\beta+i\gamma \\ 1<\gamma\leqslant T}} x^\rho = -\frac{T}{2\pi}\Lambda(x)\mathbf{1}_{\mathbb{Z}}(x) + O_x(\log T)$$

Gonek (1993) gave a uniform version with an explicit error in terms of x and T

To compare $L(s,\chi_1)$ and $L(s,\chi_2)$, one needs a variant of Gonek's theorem for arbitrary Dirichlet L-functions:

THEOREM. *Let χ be a primitive Dirichlet character mod q. Uniformly for $x>1$ and $T_2>T_1>1$, we have*

$$\sum_{\substack{\rho=\beta+i\gamma \\ T_1<\gamma<T_2}} x^\rho = -\frac{T_2-T_1}{2\pi}\Lambda(x)\chi(x)\,\mathbf{1}_{\mathbb{Z}}(x)+O(E),$$

where the sum runs over the nontrivial zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ for which $T_1<\gamma<T_2$,

$$E := x\log 2x \log_2 3x + x\log 2x\cdot\min\left\{\frac{T_2}{x},\frac{1}{\langle x\rangle}\right\} + x\log 2qT_2\cdot\log_2 3x,$$

and $\langle x\rangle$ is the distance from x to the nearest prime power other than x itself.

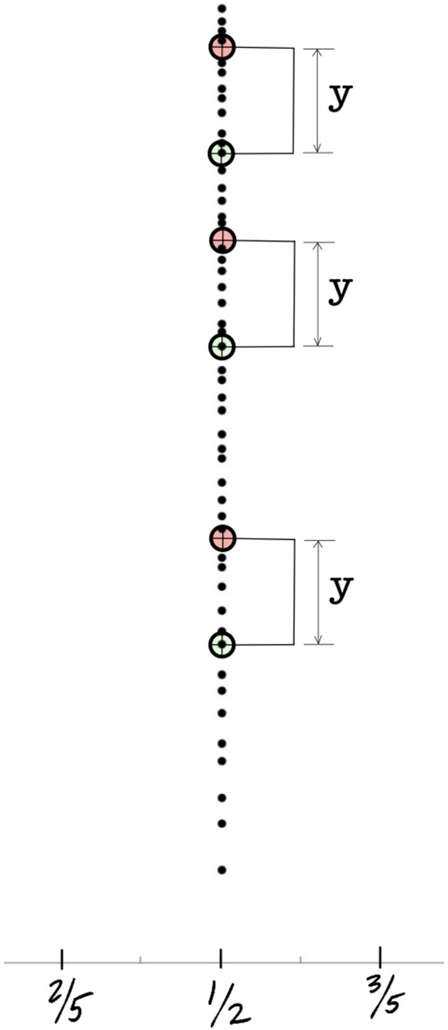
SHIFTING THE ORDINATES OF ZEROS OF THE RIEMANN ZETA FUNCTION

Let $y \neq 0$ and $C > 0$. Under the Riemann Hypothesis, for all large T we have

$$\zeta(\tfrac{1}{2} + i\gamma) = 0 \qquad \text{and} \qquad \zeta(\tfrac{1}{2} + i(\gamma + y)) \neq 0$$

for at least one γ in the interval $[T, T(1+\varepsilon)]$, where

$$\varepsilon := T^{-C/\log\log T}.$$



Theorem (B- 2024). *Assume RH. For any $y \neq 0$ and $A > 0$, there are numbers $T_0 = T_0(y, A) > 0$ and $\Theta = \Theta(A) > 0$ for which the following holds. Given T_1, T_2 satisfying $2T_1 > T_2 > T_1 > T_0$, let*

$$\Delta := (T_2 - T_1), \qquad \mathbf{T} := \tfrac{1}{2}(T_1 + T_2), \qquad \mathscr{L} := \exp\left(\frac{\log \mathbf{T}}{\log\log \mathbf{T}}\right).$$

Then, for every prime number x in the range

$$\mathbf{T}\mathscr{L}^{-\Theta} < x < \mathrm{e}^{\pi/|y|} \cdot \mathbf{T}\mathscr{L}^{-\Theta},$$

we have

$$\sum_{\substack{\rho = \frac{1}{2} + i\gamma \\ T_1 < \gamma < T_2}} x^{\rho}\zeta(\rho + iy) = \tfrac{1}{2\pi}(x^{-iy} - 1)\Delta \log \mathbf{T} + O\big(\Delta \log \mathscr{L} + \mathbf{T}\mathscr{L}^{-A}\big), \qquad ★$$

where the sum in ★ runs over zeros of $\zeta(s)$ (each zero is summed according to its multiplicity), and the implied constant in ★ depends only on y and A.

Sketch of the proof

By Cauchy's theorem (and T_0 large) we have

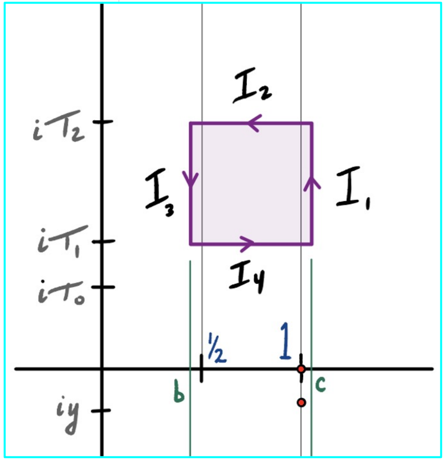
$$\sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ T_1<\gamma<T_2}} x^\rho \zeta(\rho+iy) = \frac{1}{2\pi i}\left(\int_{c+iT_1}^{c+iT_2} + \int_{c+iT_2}^{b+iT_2} + \int_{b+iT_2}^{b+iT_1} + \int_{b+iT_1}^{c+iT_1}\right)\mathcal{D}_y(s)x^s\,ds$$

$$= I_1 + I_2 + I_3 + I_4$$

$$\mathcal{D}_y(s) := \frac{\zeta'}{\zeta}(s)\zeta(s+iy)$$

$$b := \frac{1}{2} - \frac{1}{\log\log \mathbf{T}}$$

$$c := 1 + \frac{1}{\log x}$$



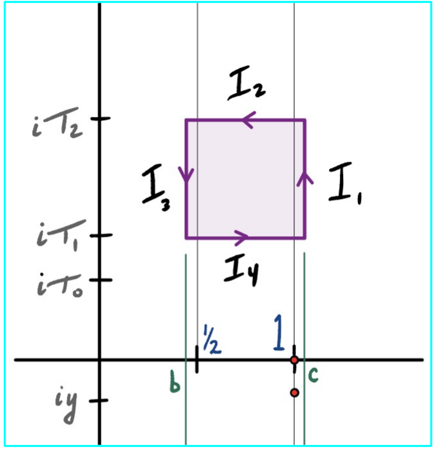
For I_1, we use the series representation

$$\mathcal{D}_y(s) = \sum_n D_y(n) n^{-s} \qquad \text{with} \quad D_y(n) := -\sum_{ab=n} \Lambda(a) b^{-iy}$$

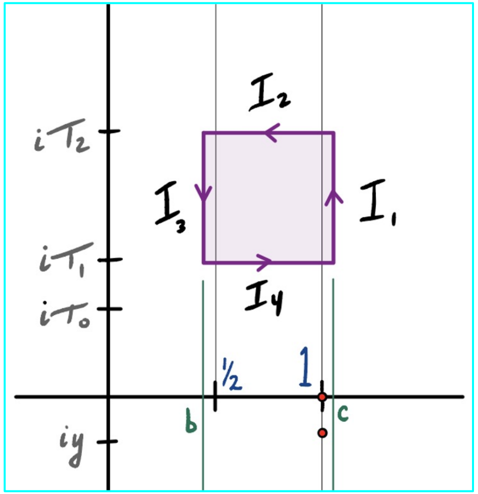
to derive the estimate

$$I_1 = \frac{1}{2\pi} \int_{T_1}^{T_2} \mathcal{D}_y(c+it,\chi)\, x^{c+it}\, dt = \frac{x^c}{2\pi} \sum_n \frac{D_y(n)}{n^c} \int_{T_1}^{T_2} \left(\frac{x}{n}\right)^{it} dt$$

$$= -\frac{\Delta}{2\pi} \log x + O(x \log^2 x)$$



Using the expansion



$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho=\beta+it \\ |\gamma-t|<1}} \frac{1}{s-\rho} + O(\log|t|)$$

we have

$$I_2 = \frac{1}{2\pi i} \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ |\gamma-T_2|<1}} \int_{c+iT_2}^{b+iT_2} \frac{\zeta(s+iy)x^s\, ds}{s-\rho} + O(\texttt{error})$$

and the rest is bounded using a method of Gonek.

Similarly for I_4.

Using the functional equations

- $\zeta(s) = \zeta(1-s)\mathcal{X}(s)$
- $\frac{\zeta'}{\zeta}(s) = -\frac{\zeta'}{\zeta}(1-s) + \log\pi - \frac{1}{2}\psi\left(\frac{1}{2}s\right) - \frac{1}{2}\psi\left(\frac{1}{2}(1-s)\right)$

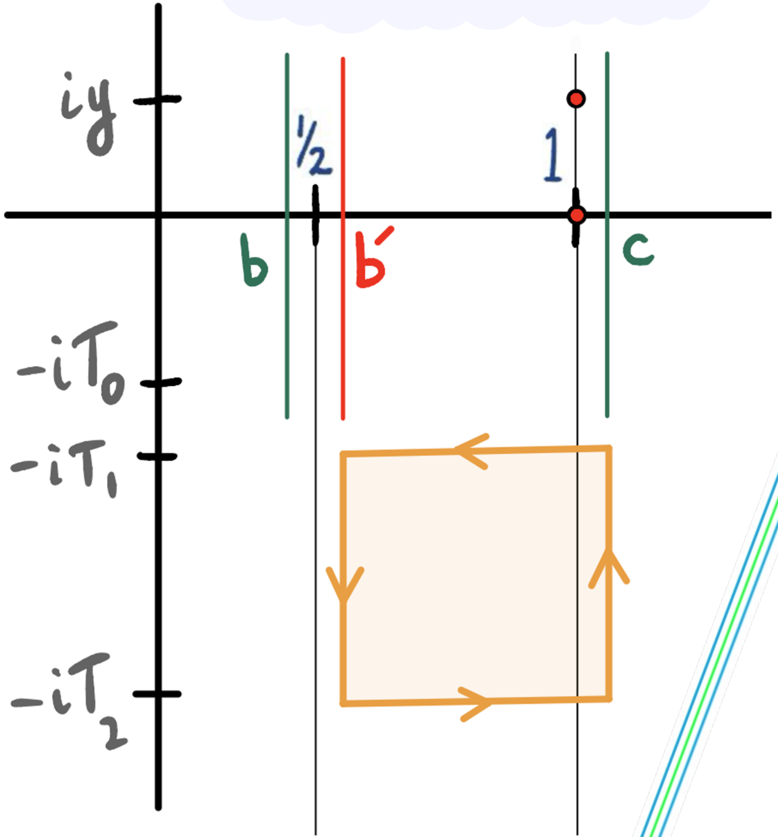
followed by the change of variables $s \mapsto 1-s$, the term I_3 equals

$$\frac{-1}{2\pi i}\int_{b'-iT_2}^{b'-iT_1}\left\{-\frac{\zeta'}{\zeta}(s)+\log\pi-\frac{1}{2}\psi\left(\frac{s}{2}\right)-\frac{1}{2}\psi\left(\frac{1-s}{2}\right)\right\}\zeta(s-iy)\mathcal{X}(1-s+iy)x^{1-s}\,ds$$

where

$$b' := 1-b = \frac{1}{2}+\frac{1}{\log\log\mathbf{T}}.$$

New contour



Shifting the line of integration back to $\sigma = c$, we have

$$I_3 = J_1 + J_2 + J_3 + O(\texttt{acceptable error}),$$

where . . . .

$$J_1 := \frac{x}{2\pi i}\int_{c-iT_2}^{c-iT_1} \frac{\zeta'}{\zeta}(s)\zeta(s-iy)\mathcal{X}(1-s+iy)x^{-s}\,ds,$$

$$J_2 := \frac{-x\log\pi}{2\pi i}\int_{c-iT_2}^{c-iT_1} \zeta(s-iy)\mathcal{X}(1-s+iy)x^{-s}\,ds,$$

$$J_3 := \frac{x}{4\pi i}\int_{c-iT_2}^{c-iT_1} \left\{\psi\left(\tfrac{s}{2}\right)+\psi\left(\tfrac{1-s}{2}\right)\right\}\zeta(s-iy)\mathcal{X}(1-s+iy)x^{-s}\,ds.$$

These integrals are estimated using the stationary phase method, taking into account the estimate

$$\mathcal{X}(1-\sigma+it) = \mathrm{e}^{\pi i/4}\exp\left(-it\log\left(\frac{|t|}{2\pi\mathrm{e}}\right)\right)\left(\frac{|t|}{2\pi}\right)^{\sigma-1/2}\{1+O_{\mathcal{I}}(|t|^{-1})\}.$$

A key ingredient is the following

Lemma. *Assume RH. Let* $y \neq 0$, $t > t' \geqslant 10$, *and put* $\Delta' := t - t'$. *For any* κ *in the range* $\mathrm{e} \leqslant \kappa \leqslant t/\mathrm{e}$, *we have*

$$\sum_{t'<n\leqslant t} D_y(n)n^{iy} \ll \Delta'\{|y|^{-1}+(\kappa/t)^{1/2}\log^2(t/\kappa+|y|)+\log\kappa\}+t/\kappa+(t\kappa)^{1/2}\log^2 t$$

which uses a result of Banks and Sinha:

$$\mathrm{RH} \Longleftrightarrow \sum_{n\leqslant x}\Lambda(n)n^{iy} = \frac{x^{1+iy}}{1+iy}+O\big(x^{1/2}\log^2(x+|y|)\big) \qquad (y\in\mathbb{R},\ x\geqslant 2)$$

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$