Unconditional comparative prime number theory over function fields

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June 21, 2024

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Comparative Prime Number Theory Symposium, Vancouver, BC

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- **Littlewood, 1914**:

$$\pi(x;4,3) - \pi(x;4,1) = \Omega_{\pm}\left(x^{1/2}\frac{\log\log\log x}{\log x}\right).$$

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- **Conjecture (Knapowski-Turán, 1962):**

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- **Rubinstein-Sarnak, 1994 :** If $L(s,\chi_4)$ satisfies GRH and LI (Linear Independence),

$$\delta(\mathcal{P}_{4;3,1}) := \lim_{X \to +\infty} \frac{1}{\log X} \int_2^X \mathbf{1}_{\mathcal{P}_{4;3,1}}(t) \frac{\mathrm{d}t}{t}$$

  exists and $\delta(\mathcal{P}_{4;3,1}) \approx 0,9959\ldots$

Rubinstein and Sarnak's results

If the Dirichlet characters modulo q statisfy GRH and LI then:

- If $a \equiv \square \bmod q$ and $b \equiv \square \bmod q$, or if $a \equiv \boxtimes \bmod q$ et $b \equiv \boxtimes \bmod q$ then $\delta(\mathcal{P}_{q;a,b}) = \frac{1}{2}$.

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- If $a \equiv \boxtimes \bmod q$ and $b \equiv \square \bmod q$ then $\frac{1}{2} < \delta(\mathcal{P}_{q;a,b}) < 1$.

- If q is of the form p^α or $2p^\alpha$, then $\frac{1}{2} < \delta(\mathcal{P}_{q;NR,R}) < 1$, where

$$\mathcal{P}_{q;NR,R} := \{x \geq 2 \mid \pi(x;q,NR) > \pi(x;q,R)\},$$

$$\pi(x;q,R) = \#\{p \leq x \mid p \equiv \square \bmod q\}$$

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The LI hypothesis

Explicit formula:

$$\frac{\pi(e^x;q,a)-\pi(e^x;q,b)}{e^{x/2}/x} = \#\sqrt{\{b\}} - \#\sqrt{\{a\}} + \sum_{\chi\in X_q} \overline{\chi(b)-\chi(a)} \sum_{\gamma_\chi} \frac{e^{i\gamma_\chi x}}{\frac{1}{2}+i\gamma_\chi} + O\left(\frac{1}{x}\right).$$

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Conjecture (LI).

The (multi)set $\bigcup_{\chi\in X_q}\left\{\gamma\geq 0 \mid L\left(\frac{1}{2}+i\gamma,\chi\right)=0\right\}$ is linearly independent over $\mathbb{Q}$.

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The **Kronecker-Weyl equidistribution theorem** tells us that $e^{i\gamma_j x}$ behave like **independent** uniform random variables on the circle.

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- Many quantities relevant to prime number theory have also been considered (point-counting over elliptic curves (**Fiorilli**), Mertens theorems (**Lamzouri**), weighted Möbius sums (**Akbary-Ng-Shahabi**), "Fouvry's bias" (**Devin**), etc.)

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- Weakening of GRH or LI (works of **Martin-Ng**, **Devin**, B.).
- **Cha** (and later **Cha-Im**) adapted the Rubinstein-Sarnak framework to **function fields**.

The canonical table

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Irreducible polynomial races

- Let $M \in \mathbb{F}_q[T]$ be non-constant and $A \in \mathbb{F}_q[T]$ coprime with M. Then

$$\pi(n; M, A) := \#\{P \in \mathbb{F}_q[T] \text{ irreducible} \mid \deg P \leq n, P \equiv A \bmod M\} \underset{n \to +\infty}{\sim} \frac{q^n}{\varphi(M)n}.$$

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and, if it exists, $d(\mathcal{P}_{M;\boxtimes,\square}) := \lim_{X\to+\infty} \frac{1}{X}\#(\mathcal{P}_{M;\boxtimes,\square} \cap [\![1, X]\!])$ its natural density.

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Theorem (Cha, 2008).

Let $M \in \mathbb{F}_q[T]$ be irreducible. Assume LI_π for the zeroes of the Dirichlet L-functions modulo M. Then $d(\mathcal{P}_{M;\boxtimes,\square})$ exists and one has

$$1/2 < d(\mathcal{P}_{M;\boxtimes,\square}) < 1.$$

The hypothesis LI_π

Theorem (Weil, 1940).

For each primitive Dirichlet character χ modulo $M \in \mathbb{F}_q[T]$, the function

$$L(s,\chi) = \sum_{A \in \mathbb{F}_q[T]} \frac{\chi(A)}{|A|^s} = \sum_{A \in \mathbb{F}_q[T]} \frac{\chi(A)}{q^{s \deg A}}$$

is a polynomial in $u := q^{-s}$ with integer coefficients:

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Conjecture (LI_π).

The (multi)set $(\{\theta_j(\chi) \mid \chi \in X_M^*, 1 \leq j \leq M(\chi)\} \cap (0,\pi)) \cup \{\pi\}$ is linearly independent over $\mathbb{Q}$.

About LI_π

- When $M \in \mathbb{F}_q[T]$ is squarefree, there exists a unique primitive quadratic character χ_M modulo M (Legendre symbol when M is irreducible).
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  - Example (**Cha**): $p = 5, M = T^5 + 3T^4 + 4T^3 + 2T + 2$ irreducible. Then $\mathcal{L}(u, \chi_M) = 25u^4 - 25u^3 + 15u^2 - 5u + 1$ with $\alpha_1 = \sqrt{5}e^{\frac{2i\pi}{5}}, \alpha_2 = \sqrt{5}e^{\frac{4i\pi}{5}}$ and we have $d(\mathcal{P}_{M;\boxtimes,\square}) \approx 40\% < \frac{1}{2}$.

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  - Example (**Devin-Meng**): $q = 9$, $M = T^4 + 2T^3 + 2T + a^7$ where $\mathbb{F}_9 = \mathbb{F}_3(a)$. Then $\mathcal{L}(u, \chi_M) = (1 - 3u)^2$ and we have $d(\mathcal{P}_{M;\boxtimes,\square}) = 1$.

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- We would like to show that LI_π still holds for "most" L-functions $L(s, \chi_M)$. There are partial results of **Kowalski** (2008) in certain one-parameter families of polynomials M which are **not** irreducible.

Some notations

- From now on, $\mathcal{H}_n(\mathbb{F}_q) := \{f \in \mathbb{F}_q[T] \mid f \text{ monic square-free of degree } n\}$ and for $f \in \mathcal{H}_n(\mathbb{F}_q)$, χ_f is the unique primitive quadratic character modulo f.

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- When f is irreducible, it is exactly the sign of $\pi(n; f, \square) - \pi(n; f, \boxtimes)$!

First results

Theorem (B.-Devin-Keliher-Li, 2024).

Let q be a power of p an odd prime and $n \geq 3$. Then

$$\frac{1}{\#\mathcal{H}_n(\mathbb{F}_q)}\#\{f \in \mathcal{H}_n(\mathbb{F}_q) \mid L(s,\chi_f) \text{ doesn't satisfy } \mathsf{LI}_\pi\} \begin{cases} \ll \frac{p}{q} \text{ if } g = 1 \\ \ll_p \frac{\log q}{q^{1/12}} \text{ if } g = 2 \\ \ll_{p,g} \frac{(\log q)^{1-\delta_g}}{q^{\varepsilon_g}} \text{ if } g \geq 3, \end{cases}$$

where $\delta_g \underset{g\to+\infty}{\sim} \frac{1}{8g}$ and $\varepsilon_g = \frac{1}{4g^2+2g+4}$.

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- **Counting:** There are $\ll p$ supersingular elliptic curves over $\overline{\mathbb{F}_q}$, and we need to count how many different $f \in \mathcal{H}_3(\mathbb{F}_q)$ or $f \in \mathcal{H}_4(\mathbb{F}_q)$ give rise to isomorphic elliptic curves over $\overline{\mathbb{F}_q}$

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- For higher genus, the main steps are the same but are much more complicated.

Sketch of proof when $g \geq 2$

- **Step 1 ("Geometric" condition):** If $\mathcal{L}(u, \chi_f)$ doesn't satisfy LI_π, then the Galois group G of $\mathcal{L}(u, \chi_f)$ is not maximal $\subsetneq W_{2g} = \mathfrak{S}_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$ (**Girstmair's method**).

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- **Step 3 ("Counting"): Kowalski's large sieve for Frobenius** and a trick due to **Chavdarov** provide an upper bound of the form

$$\ll_{p,g} H_1^{-1} + H_2^{-1} + H_3^{-1} + H_4^{-1},$$

where each H_i is given by a sum of cardinalities of appropriate sets of polynomials $P \in \mathbb{F}_\ell[T], \ell \neq 2, p$ prime, satisfying properties related to Step 2.

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- For the case $g = 2$, we get an improvement thanks to a result of **Ahmad-Shparlinski**: if LI_π fails then the Jacobian of C_f splits over $\overline{\mathbb{F}_q}$.

Failure of LI_π is not the end of the story

- Example (**Cha**): $p = 3$, $M = T^3 + 2T + 1$ irreducible. Then $\mathcal{L}(u, \chi_M) = 3u^2 - 3u + 1 = \left(1 - \sqrt{3}e^{\frac{i\pi}{6}}\right)\left(1 - \sqrt{3}e^{\frac{-i\pi}{6}}\right)$ and we have $d(\mathcal{P}_{M;\boxtimes,\square}) \approx 58,3\% > \frac{1}{2}$.

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- We want to identify "pathologic" configurations that are not necessarily implied by the failure of LI_π: **complete bias**, **reversed bias** and **lower order bias**.

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- We let

$$\Delta_f(n) := \left(m_0(\chi_f)+\tfrac{1}{2}\right) + \left(m_\pi(\chi_f)+\tfrac{1}{2}\right)(-1)^n + \sum_{\theta_j\neq 0,\pi} m_{\theta_j}(\chi_f)e^{in\theta_j(\chi_f)}.$$

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- Under LI_π, we have $1/2 < d(\Delta_f(n) > 0) < 1$.

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Theorem (B.-Devin-Keliher-Li, 2024).

We have

$$\frac{1}{\#\mathcal{H}_n(\mathbb{F}_q)}\#\{f \in \mathcal{H}_n(\mathbb{F}_q) \mid \Pi(n;\chi_f) \text{ exhibits a complete bias}\} \ll_{g,p} \frac{\log q}{q^{2\varepsilon_g}}$$

where $\varepsilon_g = \frac{1}{4g^2+2g+4}$.

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- **Step 1:** If $d(\Delta_f > 0) = 1$, then $d(\Delta_f(2n) > 0) = d(\Delta_f(2n+1) > 0) = 1$, and thanks to a variance inequality, we show that $m_0(\chi_f) > m_\pi(\chi_f)$ (and in particular q is a square).

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- **Step 3:** We use the previous large sieve method to reduce the problem to counting

$$\left\{P \in \mathbb{F}_\ell[T] \text{ monic} \mid \deg P = 2g, P(X) = q^{-g} X^{2g} P\left(\frac{q}{X}\right), P(\sqrt{q}) = 0\right\}$$

for all $\ell \neq 2, p$.

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for every prime $\ell \neq 2, p$.

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We have

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where $\varepsilon_g = \frac{1}{4g^2+2g+4}$ and $\delta_g \underset{g\to+\infty}{\sim} \frac{7}{24g} > \frac{1}{4g}$.

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- **Step 1:** If $d(\Delta_f \leq 0) > \frac{1}{2}$ then the distribution of the values of Δ_f is not symmetric with respect to its mean value $m_0(\chi_f) + \frac{1}{2} > 0$, so the torus generated by $\{(n\pi, n\theta_1(\chi_f), \ldots, n\theta_g(\chi_f)) \mid n \in \mathbb{N}\}$ in $(\mathbb{R}/\mathbb{Z})^{g+1}$ doesn't contain the central point $(\pi, \ldots, \pi)$.

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- **Step 3:** The quantity $(-1)^{k_0} \prod_{j=1}^{g} \alpha_j(\chi_f)_j^k \in \mathbb{Z}$, is fixed by G. This implies that the sequence Δ_f is degenerate, or G doesn't contain certain types of permutations.

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- **Step 4:** By Dedekind's theorem, this means that $\mathcal{L}(u, \chi_f)$ doesn't admit certain types of factorizations modulo large enough primes ℓ and we conclude using the large sieve and some combinatorics on polynomials over finite fields.

Thank you for your attention!