# Moments of $L$-functions in the World of Number Field Counting

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# Malle's Conjecture

Let $K$ be a number field and $G \subseteq S_n$ a transitive subgroup.

$$N(K,G;X) := \#\{L/K : [L:K]=n,\ \mathrm{Gal}(L/K)\cong G,\ |\mathrm{disc}(L/K)| \leqslant X\}$$

There exist constants $a = a(G)$, $b = b(K,G)$, and $c = c(K,G)$ such that

$$N(K,G;X) \sim cX^{1/a}(\log X)^{b-1}.$$

While the majority of cases of this conjecture are open, it is known for abelian groups, small permutation groups, and other specific families of groups.

# Moments of $L$-functions?

Cohen–Diaz y Diaz–Olivier: Let $D_4 \subseteq S_4$. Then

$$N(\mathbb{Q}, D_4; X) \sim \left( \frac{1}{2\zeta(2)} \sum_{[K:\mathbb{Q}]=2} \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \cdot \frac{2^{-r_2(K)}}{|\mathrm{disc}(K/\mathbb{Q})|^2} \right) X.$$

Thus, Malle's conjecture is true with $a = 1$ and $b = 1$ (predicted by Malle) and with $c$ a "weighted moment of $L$-functions".

**Proof Idea:** $D_4 = C_2 \wr C_2$, so any $D_4$-extension $L/\mathbb{Q}$ can be understood as a tower $L/K/\mathbb{Q}$ of $C_2$-extensions. Use strong counting results for quadratic fields to count $L/K$ then sum over the $K/\mathbb{Q}$.

Altug–Shankar–Varma–Wilson: Let "cond" denote the conductor. The number of $D_4$-extensions with $\mathrm{cond}(L/\mathbb{Q}) \leqslant X$ satisfies

$$N_{\mathrm{cond}}(\mathbb{Q}, D_4; X) \sim \frac{3}{4} \prod_p \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4}\right) \cdot X \log X.$$

Thus $a = 1$, $b = 2$, and $c$ is a convergent Euler product.

We find a weighted $L$-function moment via the following expression

$$N_{\mathrm{cond}}(\mathbb{Q}, D_4; X) \sim \left( \frac{3}{4\zeta(2)} \sum_{\substack{[K:\mathbb{Q}]=2 \\ |\mathrm{disc}(K/\mathbb{Q})| \leqslant X}} \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \cdot \frac{1}{|\mathrm{disc}(K/\mathbb{Q})|} \right) X.$$

Let $\mathrm{Hom}(G_K, G; X) = \{f \in \mathrm{Hom}(G_K, G) : |\mathrm{disc}(f)| \leqslant X\}$.

Let $\Sigma = (\Sigma_p)$ be a family of local conditions $\Sigma_p \subseteq \mathrm{Hom}(G_{K_p}, G)$.

Let $\mathrm{Hom}_\Sigma(G_K, G; X) = \{f \in \mathrm{Hom}(G_K, G; X) : \forall p,\ f_p \in \Sigma_p\}$.

**Theorem (A.-O'Dorney (2021))**

*Suppose $G$ is abelian and $\Sigma_p$ is Frobenian, translation invariant by $\mathrm{Hom}_{ur}(G_{K_p}, G)$ at all but finitely many places $p$, and contains 1. Then there exist explicit constants $a(\Sigma)$, $b(\Sigma)$, and $c(\Sigma)$ such that*

$$\#\mathrm{Hom}_\Sigma(G_K, G; X) \sim c(\Sigma) X^{1/a(\Sigma)} (\log X)^{b(\Sigma)-1}.$$

**Proof Idea:** The generating series $\sum |\mathrm{disc}(f)|^{-s}$ is a finite sum of Euler products, and therefore has a meromorphic continuation. The rightmost pole corresponds to the asymptotic growth rate.

# What about non-translation invariant local restrictions?

Next step: $\Sigma_p$ contains $\mathrm{Hom}_{ur}(G_{K_p}, G)$ for all but finitely many places $p$ is the only assumption.

Suppose we wanted to only count quadratic extensions $K/\mathbb{Q}$ for which $K_p$ is either unramified or $\mathbb{Q}_p(\sqrt{p})$?

$$\Sigma_p = \{\underbrace{1, u}_{unram}, \underbrace{\tau}_{ram}\} \subseteq \mathrm{Hom}(G_{\mathbb{Q}_p}, C_2)$$

How does $\#\mathrm{Hom}_\Sigma(G_\mathbb{Q}, C_2; X)$ grow?

How can moments of $L$-functions techniques be used to solve this class of questions?

A.-O'Dorney construction: Poisson summation

$$\sum_{f\in \mathrm{Hom}(G_{\mathbb{Q}},C_2)} \underbrace{|\mathrm{disc}(f)|^{-s}\mathbf{1}_{\Sigma}(f)}_{w_s(f)} = \sum_{h\in \mathrm{Hom}(G_{\mathbb{Q}},C_2)} \underbrace{\prod_p \left(\frac{1}{2}\sum_{f_p\in\Sigma_p} \langle f_p, h|_{G_{\mathbb{Q}_p}}\rangle p^{-\nu_p(\mathrm{disc}(f_p))s}\right)}_{\widehat{w_s}(h)}$$

We know some quick facts about the Tate pairing:

- It is perfect pairing, and
- $\mathrm{Hom}_{ur}(G_{\mathbb{Q}_p}, C_2)$ is exactly annihilated by $\mathrm{Hom}_{ur}(G_{\mathbb{Q}_p}, C_2)$

$$\sum_{h\in\mathrm{Hom}(G_{\mathbb{Q}},C_2)} \frac{\xi(h)}{2^{\omega(\mathrm{disc}(h))}|\mathrm{disc}(h)|^s} \prod_{p\nmid\mathrm{disc}(h)} \left(1+\frac{1}{2}\langle\tau, h|_{G_{\mathbb{Q}_p}}\rangle p^{-s}\right)$$

For $\xi(h)=\prod_{p|\mathrm{disc}(h)}\langle\tau, h_{G_{\mathbb{Q}_p}}\rangle\in\{\pm 1\}$.

By appropriately factoring the Euler factors we get

$$\sum_{h\in\mathrm{Hom}(G_{\mathbb{Q}},C_2)} \frac{\xi(h)}{2^{\omega(\mathrm{disc}(h))}|\mathrm{disc}(h)|^s} L(s,h)^{1/2} G_h(s).$$

Or, approximately,

$$\#\mathrm{Hom}_{\Sigma}(G_{\mathbb{Q}},C_2;X)\sim \underbrace{\frac{X}{(\log X)^{1/2}}}_{h=1} + \sum_{\substack{[K:\mathbb{Q}]=2\\ |\mathrm{disc}(K/\mathbb{Q})|\leqslant X}} L(1,K/\mathbb{Q})^{1/2}\frac{\xi(K/\mathbb{Q})G_{K/\mathbb{Q}}(1)}{2^{\omega(\mathrm{disc}(K/\mathbb{Q}))}}$$

# Thanks for coming!!