RELATIONS BETWEEN $\triangle + \cdots + \triangle + 3\triangle + \cdots + 3\triangle$ AND $\square + \cdots + \square + 3\square + \cdots + 3\square$

ZAFER SELCUK AYGIN

For non-negative integers a, b and n, let

$$t(a,b;n) = \# \left\{ (x_1, \ldots, x_a, y_1, \ldots, y_b) \in \mathbb{Z}^{a+b} \ \middle| \ n = \frac{x_1(x_1-1)}{2} + \cdots + \frac{x_a(x_a-1)}{2} + 3\frac{y_1(y_1-1)}{2} + \cdots + 3\frac{y_b(y_b-1)}{2} \right\}$$

and

$$N(a,b;n) = \#\{(x_1, \ldots, x_a, y_1, \ldots, y_b) \in \mathbb{Z}^{a+b} \ | \ n = x_1^2 + \cdots + x_a^2 + 3y_1^2 + \cdots + 3y_b^2\}.$$

By works of Bateman and Knopp & Adiga, Cooper and Han, it is known that for $(a,b) = (1,0), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0)$, and $(3,1)$ we have

$$\frac{t(a,b;n)}{N(a,b;8n+a+3b)} = \frac{2}{2^{a+b-2} + 2^{(a+b-2)/2}\cos(\pi(a+3b)/4) + 1}$$

for all $n \in \mathbb{N}$. Moreover, for $(a,b) = (8,0)$ and $(2,2)$, Baruah, Cooper and Hirschhorn has proven that

$$\frac{t(a,b;n)}{N(a,b;8n+a+3b) - N(a,b;(8n+a+3b)/4)} = \frac{2}{2^{a+b-2} + 2^{(a+b-2)/2}}$$

for all $n \in \mathbb{N}$.

Variations of such identities were observed by many mathematicians including Baruah, Bateman, Cooper, Dastovski, Hirschhorn, Knopp, Sun and Williams. However, it seems that for bigger values of a and b these ratios start to fluctuate when n varies.

In this work we investigate the limiting cases of similar ratios when $a+b$ is an even integer greater than 4, $a \geq 2$ and $b \geq 0$. We show that previously known examples are special cases of an asymptotic relation between $N(a,b;n)$ and $t(a,b;n)$. We achieve our results by finding certain modular identities which relates generating functions of $t(a,b;n)$ to $N(a,b;n)$. This is a joint work with Amir Akbary (University of Lethbridge).