Day 4

(partial) Landscape of theories & twists

2d $\mathcal{N}=(2,2)$ $\quad T^{2d}[\mathfrak{X}]$ $\quad \mathfrak{X}$ Kähler

$T^{2d}[G,V,W]$ $\quad$ "linear gauge theory"

G : reductive gp / $\mathbb{C}$

V : linear rep

$W : V/G \to \mathbb{C}$

two top'l twists $\quad Q_A$ A $\quad$ B Q_B

holomorphic twist $\quad$ H $\quad Q_H$

$$\begin{tikzcd} A \arrow[rr, leftrightarrow, "MS"] & & B \\ & H \arrow[ul] \arrow[ur, "\text{deformations}"'] & \end{tikzcd}$$

Q_H makes $\partial_{\bar z}$ exact but not ∂_z on 2d spacetime locally $\mathbb{C}_{z,\bar z}$

bulk local ops in H twist = chiral de Rham ($\mathfrak{X}$)

$Q_A = Q_H + Q_A'$ $\qquad$ Q_A' is an additional differential in the H twist

$Q_B = Q_H + Q_B'$

3d $\mathcal{N}=2$ $\quad T^{3d}[\mathcal{X}]$ $\quad \mathcal{X}$ Kahler

$T^{3d}[G, V, W, k]$ $\quad k \in H^4(BG)$-torsor

$T^{3d}[\mathfrak{g}, M^3]$ $\quad \mathfrak{g} \in ADE$, M^3 3-manifold $\quad$ (3d-3d correspondence)

only twist is holomorphic-topological $\quad$ HT

spacetime locally $\mathbb{R}_t \times \mathbb{C}_{z,\bar{z}}$

(globally: THF structure)

∂_t, $\partial_{\bar{z}}$ are Q_{HT}-exact

3d $\mathcal{N}=4$ $\quad T^{3d}[\mathcal{Y}]$ $\quad \mathcal{Y}$ hyperkahler

(algebraic symplectic)

$$T^{3d}[G, \underbrace{V = T^*\underset{y}{N} \oplus \underset{\alpha}{\mathfrak{g}}}_{T^*}, W = \langle \alpha, \mu(x,y) \rangle^{+\tau}, k=0] =: T^{3d}_{\mathcal{N}=4}\{G, N\}$$

$\underset{x}{N}$ $\sim y \cdot \alpha \cdot x$

$$A \longleftarrow HT \longrightarrow B$$

Rozansky-Witten

3d MS

(A & B fully topological

i.e. $\partial_t, \partial_{\bar{z}}, \partial_z$ are all Q_A, Q_B-exact)

3d $\mathcal{N}=4$ thy's are closely tied to geometric rep thy

- Aganagic - Okounkov (elliptic coh) $\longleftrightarrow$ HT twist $\circlearrowleft$ MS
- $\mathcal{M}^{\text{aff}}_{\text{Higgs}}[G,N] = \operatorname{Spec}(\operatorname{Ops}^{B}_{G,N})$

$$\simeq T^*N /\!\!/ G$$

$$\mathcal{M}^{\text{aff}}_{\text{Coul}}[G,N] = \operatorname{Spec}(\underset{\substack{\uparrow \\ \text{BFN}}}{\operatorname{Ops}^{A}_{G,N}})$$

$$\begin{tikzcd} \text{3d} \arrow[d, "S^1"'] \\ \text{2d} \end{tikzcd} \qquad \begin{tikzcd} & \text{HT} \arrow[dl, "S^1_{\text{hol}}"'] \arrow[d, "S^1_{\text{top}}"] \arrow[dr, "S^1_{\text{top}} + \text{rotation}"] & \\ B & H & A + \Omega\text{-backgd} \end{tikzcd}$$

$$\mathbb{C} \times [0,R] \Big/ (z,R) \sim (qz,0)$$

$$\begin{tikzcd} A \arrow[d, "S^1_{\text{top}}"] \\ A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "S^1_{\text{top}}"] \\ B \end{tikzcd}$$

4d $\mathcal{N}=1$ $\quad T^{4d}[\mathcal{X}]$ or $T^{4d}[G, V, W]$

$$\begin{array}{c} HH \text{ twist } \quad \mathbb{C}\times\mathbb{C} \\ \big\downarrow S^1_{hol} \\ 3d \text{ HT twist of } \mathcal{N}=2 \quad (\text{on loop space}) \end{array}$$

4d $\mathcal{N}=2$ $\quad T^{4d}[\mathcal{Y}]$ or $T^{4d}[G, T^*N \oplus \mathfrak{g}, W=\langle \alpha, \mu\rangle] =: T^{4d}_{\mathcal{N}=2}[G, N]$

or $T^{4d}[\mathcal{Y}, \varepsilon]$

$$\begin{array}{ccccc} HH & \longrightarrow & \boxed{\begin{array}{c} HT \\ \mathbb{C}\times\mathbb{R}^2 \\ \text{GMN} \end{array}} & \longrightarrow & A \ (\text{Donaldson}) \\ \mathbb{C}\times\mathbb{C} & & & & \end{array}$$

BPS states, line ops, surf ops, ...

$$\begin{array}{ccc} 4d\ \mathcal{N}=2 & & A \\ S^1 \big\downarrow & & \big\downarrow S^1_{top} \\ 3d\ \mathcal{N}=4 & & A \end{array}$$

$$\begin{tikzcd} & HT \arrow[dl, "S^1_{hol}"'] \arrow[d, "S^1_{top}"] \arrow[dr, "S^1_{top} + \text{rotate}"] & \\ B & HT & A + \Omega \text{ def} \end{tikzcd}$$

Algebraic objects in 3d HT twist

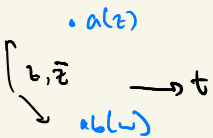
$V = \mathbb{C}$

Example : $T^{3d}[\mathbb{C}]$

bulk local ops $\mathcal{V}$ a vertex algebra

commutative (nonsingular OPE)

Poisson



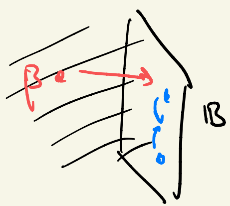
$\mathcal{V} = \mathbb{C}[\underbrace{T^*[1]\,\mathbb{C}}\llbracket z \rrbracket]$

$= \langle\langle x(z),\ \psi(z) \rangle\rangle$

ie local ops are $\partial_n x(z) \quad \partial_n \psi(z)$

$\{x, \psi\}(z) = 1$

boundary cond^s form a set Bdy (w/ $\oplus$)



for any $\mathbb{B} \in \mathrm{Bdy}$,

local ops on $\mathbb{B}$ $\mathcal{V}_{\mathbb{B}}$

are a potentially non-comm. vertex alg.

(singular OPE)

$\mathcal{V}_{D(0)} = \langle\langle x(z) \rangle\rangle$

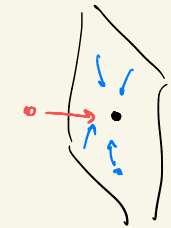
$\mathcal{V}_{D(1)} = \langle\langle \psi(z) \rangle\rangle$

bulk-bdy maps

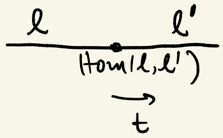
$\beta_{\mathbb{B}} : \mathcal{V} \to Z(\mathcal{V}_{\mathbb{B}})$ map of vertex alg^s

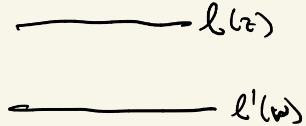
$\beta^{der}_{\mathbb{B}} : \mathcal{V} \xrightarrow{\sim} \mathrm{End}^{\bullet}_{\mathcal{V}_{\mathbb{B}}\text{-mod}}(\mathcal{V}_{\mathbb{B}})$

if $\mathbb{B}$ is big enough



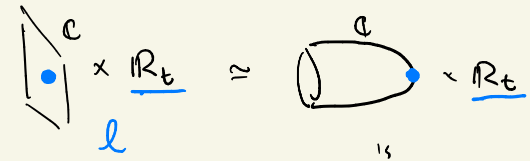
New: category of line operators $\mathbb{L}\mathrm{ines}$ $\qquad$ $T^{3d}[\mathbb{C}]$





$\mathbb{L}\mathrm{ines} = D^b\mathrm{Coh}(\mathbb{C}((z)))$

why?



expect

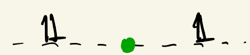
"OPE of lines"

$$\ell(z)\otimes\ell'(w) \simeq \bigoplus_{n\geq -N}\frac{1}{(z-w)^n}\,\ell_n(w)$$

2d B-model target $\mathbb{C}((z))$

ℓ is a b.c.

There's a $\otimes$-identity $\mathbb{1}$ trivial line

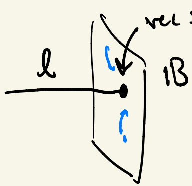


$$\mathcal{V} = \mathrm{End}^\bullet(\mathbb{1})$$

$$\mathbb{1} = \mathcal{O}_{\mathbb{C}[[z]]}$$

Exercise: recover $\mathcal{V} = \mathrm{End}^\bullet_{D^b\mathrm{Coh}(\mathbb{C}((z)))}(\mathcal{O}_{\mathbb{C}[[z]]})$

- For any $\mathbb{B}$



$$F : \mathbb{L}\mathrm{ines} \xrightarrow{\sim} \mathcal{V}_{\mathbb{B}}\text{-mod}$$

if $\mathbb{B}$ is big enough

Don't know $\mathbb{L}\mathrm{ines}$ or $\underset{\text{(bulk)}}{\mathcal{V}}$ in general gauge theories!

For G, $V = 0$, k

$\mathrm{Coh}^{G_k}(pt) = \mathrm{Rep}(G_k)$

Guess $T[G, V, W, k] \rightsquigarrow \mathbb{L}\mathrm{ines} \sim MF^{G_k}(V((z)), \oint W)$

loop gp of G extended at level k

3d $\mathcal{N}=4$ $\quad \mathbb{L}\mathrm{ines}^A$, $\mathbb{L}\mathrm{ines}^B$ become braided $\otimes$ cats

deformations of $\mathbb{L}\mathrm{ines}^{HT}$

G, N

$$\mathbb{L}\mathrm{ines}^A \simeq D\text{-mod}\left(N((z))/G((z))\right)$$

$$\simeq D\text{-mod}^{G((z))}(N((z))) \qquad \mathbb{C}[\mathcal{M}_{coul}] = \mathrm{End}_{\mathbb{L}\mathrm{ines}^A}(\mathbb{1})$$

$$\mathbb{L}\mathrm{ines}^B \simeq MF^{G((z))}\Big(T^*N((z)) \times \underset{\text{punc}^{d}\text{ disc}}{\mathrm{Conn}^A(D^\times)}, \oint x\, \partial_A y\Big)$$