Day 3

2d topologically twisted QFT

Recall: Poisson algebra of bulk local ops $\xleftarrow[Q\text{-coh}]{}$ E_2 algebra
(+ higher L_∞ ops)

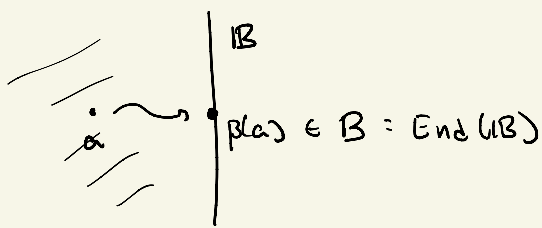
Ops

For any boundary condition $\mathbb{B} \in \mathrm{Bdy}$ there's an associative alg
(+ higher A_∞ ops)

$$B = \mathrm{End}_{\mathrm{Bdy}}(\mathbb{B})$$

Underived bulk-bdy map

$$\beta_{\mathbb{B}} : \mathrm{Ops} \longrightarrow B$$



respects the product $\Rightarrow$ factors through $Z(B)$.

bulk $\{\ ,\ \}$ preserves $\ker \beta_{\mathbb{B}}$ exercise

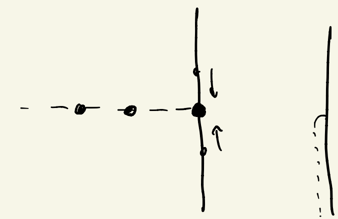
algebraically, b.c. are isotropic.

<u>Derived bulk-bdy map</u> $\beta^{der} : \text{Ops} \xrightarrow{\sim} HH^\bullet(B)$ ← assoc alg. of bdy local ops

For any $\mathbb{B} \in \mathbb{B}dy$, let $B := \text{End}_{\mathbb{B}dy}(\mathbb{B})$ Hochschild cohomology, ie derived center

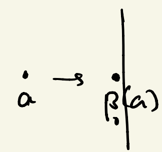
If $\mathbb{B}$ is large enough (i.e. generates $\mathbb{B}dy$) $\Rightarrow$ $HH^\bullet(B) =: HH^\bullet(\mathbb{B}dy)$

then β^{der} is an isomorphism.

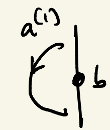


$$\beta^{der}_\bullet = \bigoplus_{n \geq 0} \beta_n$$

β_0 = ordinary b.b. map

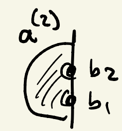


$\beta_1(a) \in \text{Hom}_{\mathbb{C}}(B, B)$

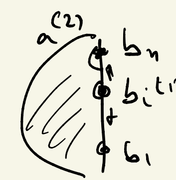


$= \beta_1(a)(b)$

$\beta_2(a) \in \text{Hom}_{\mathbb{C}}(B^{\otimes 2}, B)$

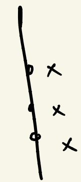


$\beta_n(a) \in \text{Hom}_{\mathbb{C}}(B^{\otimes n}, B)$



Example: B-model to $\mathbb{C}$ $\qquad D^{(0)}$ $\qquad \Psi| = 0$

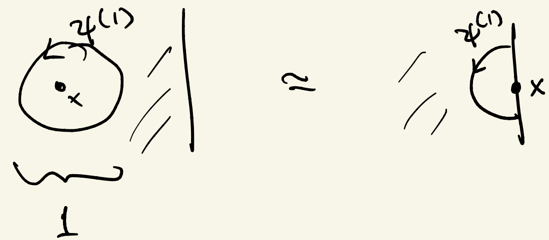
X "free"



ψ, x

$\beta_0(\psi) = 0 \qquad \beta_0(x) = x$

$\beta_1(\psi)(x) = 1 \qquad \beta_1(x) = 0$



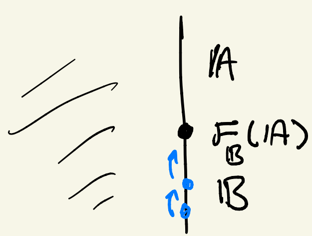
i.e. $\beta_1(\psi)$ acts as ∂_x on $B = \mathbb{C}[x]$

Generators: Math: for any object $\mathbb{B} \in \mathbb{B}dy$ there's a $\overset{(A_\infty)}{\text{functor}}$

$$F_{\mathbb{B}} : \mathbb{B}dy \to B\text{-mod}$$

$$\mathbb{A} \mapsto \mathrm{Hom}^{\bullet}_{\mathbb{B}dy}(\mathbb{B}, \mathbb{A})$$

$B = \mathrm{End}(\mathbb{B})$



I say $\mathbb{B}$ generates if $F_{\mathbb{B}}$ is an equivalence of derived / A_∞ cats.

Almost sufficient: $\mathrm{Hom}^{\bullet}_{\mathbb{B}dy}(\mathbb{B}, \mathbb{A}) \neq \phi \ \forall\ \mathbb{A} \in \mathbb{B}dy$

Example B-model to $\mathbb{C} \circlearrowleft \mathbb{C}^*$

$\mathcal{D}^{(0)} \sim \mathcal{O}$ generates!

End $\simeq \mathbb{C}[x]$

$\mathcal{D}^{(1)} \sim$ skyscraper at 0 $\mathcal{O}_0$

also generates IF restrict to $\mathbb{C}^*$ equivt sheaves

End $\simeq \mathbb{C}[\psi]$

$$D^b \mathrm{Coh}(\mathbb{C}) \simeq D^b\, \mathbb{C}[x]\text{-mod}$$

$$\underbrace{D^b\, \mathbb{C}[x]\text{-gr-mod}}_{\text{(equiv) coh sheaves}} \simeq D^b\, \mathbb{C}[\psi]\text{-gr-mod}$$

More interesting B-model

target $\mathcal{X}$, $W : \mathcal{X} \to \mathbb{C}$ algebraic

$\mathbb{Z}$ graded if give functions on $\mathcal{X}$ nonzero coh. degree st. $|W| = 2$

BV action (locally) $\int_{\mathbb{R}^2} \Psi_i \, dX^i + W(X)$

Bulk local ops $\simeq$ $\mathbb{C}[\mathrm{DCrit}(W)]$ deformation of $\mathbb{C}[T^*[1]\mathcal{X}]$

same algebra + differential Q $\quad$ $\mathbb{C}[\psi_i, x^i]$, $\psi_i \sim \frac{\partial}{\partial x^i}$

s.t. $Q(x^i) = 0$

$Q(\psi_i) = \frac{\partial}{\partial x^i} W(x)$

Exercise: derive from BV action. Compute $QX = \{S, X\}_{BV}$ etc.

Example $\mathcal{X} = \mathbb{C}^3_{x,y,z}$ $\qquad W = XYZ$ $\qquad$ Bulk: $\mathbb{C}[x, y, z, \psi_x, \psi_y, \psi_z]$

$Q\psi_x = yz$

$Q\psi_y = xz$

$Q\psi_z = xy$

Note $\{S, S\}_{BV} = \int_{\mathbb{R}\times\mathbb{R}_+} d(\Psi_i X^i) + dW$

$= \int_{\mathbb{R}\times\{0\}} \Psi_i X^i + W$

Simplest b.c are Lagrangians in $T^*[1]\mathbb{C}^3$ w/ $\boxed{W = 0}$

Three good b.c. :

$\cancel{\mathcal{D}^{(0)} \quad \psi_x| = 0 \quad \psi_y| = 0 \quad \psi_z| = 0 \quad W| \neq 0}$

$\mathcal{D}^{(1)} \quad X| = 0 \quad \psi_y| = \psi_z| = 0$

$$\mathrm{End}(\mathcal{D}^{(1)}) = \boxed{\mathbb{C}[\psi_x, y, z] \text{ with } Q\psi_x = yz = \frac{\partial W}{\partial x}\Big|}$$

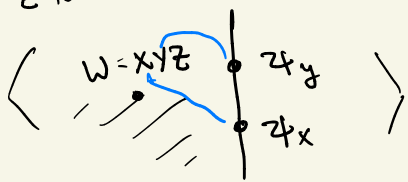
both generate (w/ C^0 equiv.)

$\mathcal{D}^{(2)} \quad X| = 0 \quad Y| = 0 \quad \psi_z| = 0$

tracks Q for N.C.

$$\mathrm{End}(\mathcal{D}^{(2)}) = \boxed{\mathbb{C}[\psi_x, \psi_y, z] \text{ with } \psi_x\psi_y + \psi_y\psi_x = z \text{ i.e. } [\psi_x, \psi_y] = \frac{\partial W}{\partial x \partial y}\Big|}$$

set Z to nonzero constant



$$\psi_x \psi_y e^{-S} = \langle \psi_x \psi_y \rangle_{class} - \boxed{\langle \psi_x \psi_y W \rangle_{class}} + \frac{1}{2}\langle \cdot W^2 \rangle$$

$\neq 0$

exercise!

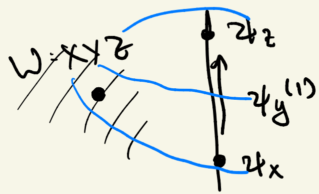
$$\langle \psi_x(0)\psi_y(1) + \psi_y(0)\psi_x(1) \rangle = Z$$

$\mathcal{D}^{(3)}$ $\quad X| = Y| = Z| = 0$

$\mathrm{End}(\mathcal{D}^{(3)}) \cong \mathbb{C}[\psi_x, \psi_y, \psi_z]$ $\qquad$ No Q since $\partial W| = 0$

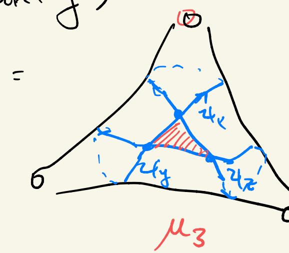
A_∞ algebra $\qquad$ No NC since $\partial^2 W| = 0$

w/ $\mu_3(\psi_x, \psi_y, \psi_z) = \dfrac{\partial W}{\partial x \partial y \partial z}\Big|$

$\Big\langle$ 

$\Big\rangle \neq 0$.

Mirror of $W = XYZ$ is $\mathrm{Fuk}(Y)$

$Y =$ 

Bdy $\simeq$ $\mathrm{End}(\mathcal{D}^{(i)})$-gr mod $\quad \forall\, i = 1, 2, 3$

($\mathbb{C}^*$ equiv.) $\qquad$ A_∞

X	Y	Z
1	1	-2

$|W| = 0$

	X	Y	Z	ψ_x	ψ_y	ψ_z
coh degree:	1	1	0	0	0	1

this

$\mathrm{Coh}(\mathbb{C}^3)$ means $\mathbb{C}[x, y, z]$-mod

as a graded algebra

Standard Math answer for $\mathbb{B}dy = MF(\mathcal{X}, W)$

$\mathcal{X}, W$

objects are complexes of sheaves $\mathcal{E}^\bullet$ on $\mathcal{X}$

w/ an endo $\varphi_\mathcal{E} : \mathcal{E}^\bullet \circlearrowleft$

st $|\varphi_\mathcal{E}| = 1$ and $\boxed{\underset{\deg 2}{\varphi_\mathcal{E}^2} = \underset{2}{W} \cdot}$

$$\mathrm{Hom}^\bullet((\mathcal{E}^\bullet, \varphi_\mathcal{E}), (\mathcal{F}, \varphi_\mathcal{F}))$$

$$= \mathrm{Hom}^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet), \quad Q\alpha = \alpha\varphi_\mathcal{E} - \varphi_\mathcal{F}\alpha$$

$\varphi_\mathcal{E}^2 = W$ cancels out $\quad Q_{bulk}^2 = \{S, S\}_{BV} = -\int_{bdy} W$

$\parallel$

Q_{bdy}^2

$$(Q_{bulk} + Q_{bdy})^2 = W - W = 0$$

In $\mathbb{Z}_2$ setting, $\quad \mathcal{E}^0 \underset{\varphi_0}{\overset{\varphi_1}{\rightleftarrows}} \mathcal{E}^1 \qquad \varphi_0\varphi_1 = \varphi_1\varphi_0 = W \cdot \mathrm{id}$

Oblomkov-Rozansky: line ops on bdy of a 3d B-model