# Branes, Quivers and BPS algebras

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## 2.7. Recapitulation

- We have argued that the derived category of coherent sheaves form a good model of branes and their bound states:
  - We found coherent sheaves associated with a stack of branes supported on subvarieties inside $\mathbb{C}^3$.
  - Non-reduced schemes have a physical interpretation in terms of turning on an expectation value for the Higgs field.
  - Complexes of sheaves can be interpreted as bound states of branes with a tachyonic field of non-trivial profile.
  - Quasi-isomorphisms then encode the processes of tachyon condensation.
- Morphisms $Hom^n(A, B)$ in the brane category correspond to massless string modes.
- $\mathrm{Hom}^n(A, B)$ can be computed as morphisms $\mathrm{Hom}(\tilde{A}, \tilde{B})$ between projective resolutions of our branes in the homotopy category (chain maps modulo chain homotopies).

## 2.8. Supersymmetric quantum mechanics

- We are now going to adapt the above tools to derive framed quivers with potential describing the low-energy dynamics of D0-branes bound to a fixed configuration of non-compactly supported branes.
- The low-energy dynamics of D0-branes is captured by a sypersymmetric gauged quantum mechanics with potential.
- Such a quantum mechanics is specified by
  - a gauge group $G$ specifying fields forming a vector multiplet,
  - a representation $M$ of the group $G$ specifying fields forming a chiral multiplet,
  - a holomorphic functions on $M$ invariant under $G$ called superpotential $W$.

  See e.g. [Ohta-Sasai (2014)] for details.
- Today, we are now going to derive this data from calculations in the derived category of coherent sheaves, see e.g. [Sharpe (2003), Aspinwall-Katz (2004), Butson-MR (in progress)].

# 2.9. The gauge node

- The pair $(G, M)$ coming from branes on a toric Calabi-Yau threefold can be encoded in terms of a framed quiver diagram.
- The gauge group $G$ is going to be generally a product of $U(n_i)$ factors, each associated to a generator of the subcategory of compactly-supported branes.
- Since all the compactly supported branes in our $\mathbb{C}^3$ example are D0-branes, we have a single node with label $n$ specifying the number of such D0-branes.
- Diagrammatically, we associate a circular node with each $U(n_i)$ factor and attach an integer $n$ to it.

$G = U(n_1) \times \dots \times U(n_m)$

$n_1$ ... $n_m$

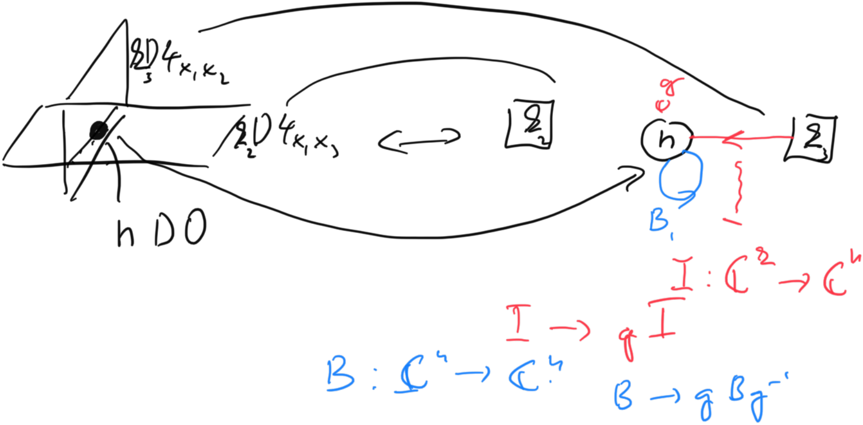
$n$ D0

$U(n)$

$n$

## 2.10. The framing node

- We associate a square (framing) node to each elementary non-compactly supported brane and attach an integer $k_j$ to it determining the number of branes in the given stack.



## 2.11. Arrows

- The representation $M$ is encoded by arrows in the quiver diagram joining different nodes.
- Each arrow is in correspondence with a factor in $M = \oplus_\alpha M_\alpha$:
- Each factor $M_\alpha$ is associated with a map $\mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$ with $n_i$ being the integer attached to the tail node and $n_j$ the integer attached to the terminal node of the arrow.
- A generator $g \in U(n_i)$ of the gauge group $G$ acts on all $M_\alpha$ associated with arrows ending at the corresponding node by multiplication from the left and on all $M_\alpha$ associated to arrows starting at the corresponding node by multiplication by $g^{-1}$ from the right.
- Physically, $(G, M)$ determine fields of the quiver QM we want to construct. In turn, such fields should arise from massless strings stretched between our branes computed be $\mathrm{Hom}(A, B)$. We are thus going to identify the arrows with generators of $\mathrm{Hom}(A, B)$.

- In order to arrive at the desired quivers, we need to:
  - Restrict to morphisms $\mathrm{Hom}^1(A, B)$ of ghost-number one since only these contribute to physical modes.
  - Shifts complexes associated D0-branes by one. As explained above, construction of a bound states requires the degree of one of the two branes to be shifted. We thus need to introduce a shifts of complexes associated D0-brane:

$$\text{D0:} \quad \overset{-4}{\mathcal{O}} \to \overset{-3}{\mathcal{O}^3} \to \overset{-2}{\mathcal{O}^3} \to \overset{-1}{\mathcal{O}}$$

$$\downarrow$$

$$\text{D0[1]:} \quad \mathcal{O} \to \mathcal{O}^3 \to \mathcal{O}^3 \to \mathcal{O}$$

$$A \to \underline{\text{D0[1]}}$$

# 2.12. D0-D0 strings

- Let us now write down generators of $\mathrm{Hom}^1(D0[1], D0[1])$. We have for example generator $b_1$ given by

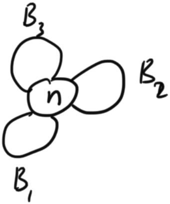
$b_1$
$b_2$
$b_3$

$$
\begin{tikzcd}[column sep=large]
 & \mathcal{O} \arrow[r, "\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}"] & \mathcal{O}^3 \arrow[r, "\begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_3 & x_1 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}"] & \mathcal{O}^3 \arrow[r, "\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1, 0, 0 \end{pmatrix}"] & \mathcal{O} \\
\mathcal{O} \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O} &
\end{tikzcd}
$$

$(0,0,1)$
$(0,1,0)$

and analogously for $b_2, b_3$.

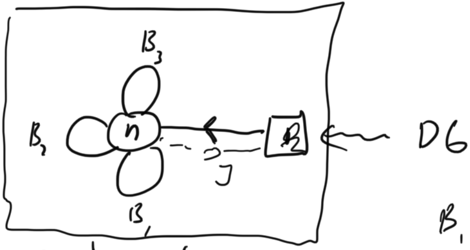
- This lead to the quiver:



## 2.13. D0-D6 strings

Hom'(D6, D0[1]):

$$\begin{tikzcd} & & & \mathcal{O} \arrow[d, "1"] \\ \mathcal{O} \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O} \end{tikzcd} = i$$




$\mathcal{O}$

$B_1 I = 0$

$B_2 I = 0$

Hom'(D0[1], D6) = $\emptyset$

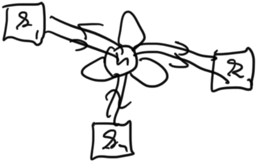
# 2.14. D0-D4 strings

Hom'(D4, D0[1])

$$\begin{tikzcd}
 & & \mathcal{O} \arrow[r] \arrow[d, "\begin{pmatrix}0\\0\\-1\end{pmatrix}"] & \mathcal{O} \arrow[d, "1"] \\
\mathcal{O} \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O}
\end{tikzcd}$$

Hom'(D4, D0):

$$\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[d, "1"] & \mathcal{O}^3 \arrow[r] \arrow[d, "\begin{pmatrix}0 & 0 & 1\end{pmatrix}"] & \mathcal{O}^3 \arrow[r] & \mathcal{O} \\
\mathcal{O} \arrow[r] & \mathcal{O} & &
\end{tikzcd}$$



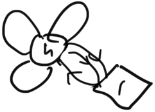

# 2.15. D0-D2 strings

$\mathrm{Hom}^1(D2, D0[1])$

$$
\begin{tikzcd}
 & \mathcal{O} \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}"] & \mathcal{O}^2 \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] & \mathcal{O} \arrow[d, "1"] \\
\mathcal{O} \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O}
\end{tikzcd}
$$

$\mathrm{Hom}^1(D0[1], D2)$

$$
\begin{tikzcd}
 & \mathcal{O} \arrow[r] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \mathcal{O}^3 \arrow[r] \arrow[d, "\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}"] & \mathcal{O}^3 \arrow[r] & \mathcal{O} \\
\mathcal{O} \arrow[r] & \mathcal{O}^2 \arrow[r] & \mathcal{O} & &
\end{tikzcd}
$$




$$
\begin{tikzcd}
 & \mathcal{O} \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \mathcal{O}^3 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}"] & \mathcal{O}^3 \arrow[r] & \mathcal{O} \\
\mathcal{O} \arrow[r] & \mathcal{O}^2 \arrow[r] & \mathcal{O} & &
\end{tikzcd}
$$

## 2.16. String modes

- For completeness, let us also write down dimensions of all $\mathrm{Hom}^n(A, B)$:

| dim $\mathrm{Hom}^n$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|---|---|---|---|---|
| D0[1]-D0[1] | 1 | 3 | 3 | 1 |
| D0[1]-D6 | | | 1 | |
| D6-D0[1] | | 1 | | |
| D0[1]-D4 | | 1 | 1 | |
| D4-D0[1] | | 1 | 1 | |
| D0[1]-D2 | 1 | 2 | 1 | |
| D2-D0[1] | | 1 | 2 | 1 |

## 2.17. Potential

- Let $x_i$ for $i \in$ arrows be generators of $\mathrm{Hom}^1(A, B)$ between various elementary branes in a given background. Any element in $\mathrm{Hom}^1(A, B)$ can be then written as a linear combination

$$\Psi = \sum_{i_k \in \text{arrows}} X_k x_i$$

where $X_k : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$ for $n_i, n_j$ ranks associated with the tail and the head of the arrow $i$. In the string-field-theory litarature, this linear combination is called the string field.

- The potential (as a function of $X_k$) is generally given by an $A_\infty$ structure $\mu_m$ of the brane category together with a trace map (related to Serre duality) $\int$ in terms of

$$\boxed{W = \sum_{k=2}^{\infty} \frac{1}{k+1} \int \mu_2(\Psi, \mu_m(\Psi, \ldots, \Psi))}$$

- Strings can mutually join and split, leading to an associative product (often called the star product $\mu_2(\alpha_1, \alpha_2) = \alpha_1 \star \alpha_2$ in the string-field-theory literature)

$$\star : \mathrm{Hom}^*(A_1, A_2) \otimes \mathrm{Hom}^*(A_2, A_3) \to \mathrm{Hom}^*(A_1, A_3)$$

- More generally, there also exist higher products

$$\underbrace{\mathrm{Hom}^*(A_1, A_2) \otimes \cdots \otimes \mathrm{Hom}^*(A_n, A_{n+1})}_{n} \to \mathrm{Hom}^*(A_1, A_{n+1})$$

forming an $A_\infty$-structure.

- Luckily, these are trivial for $\mathbb{C}^3$ and the potential is simply

$$W = \int \Psi \star \Psi \star \Psi$$

with $\star$ given by the composition of morphisms.

- Note also the symmetry of the first table above

$$\dim \mathrm{Hom}^n(D0, A) = \dim \mathrm{Hom}^{3-n}(A, D0)$$

- This is a consequence of the Serre duality stating that there exists a natural pairing

$$\mathrm{Hom}^n(D0, A) \times \mathrm{Hom}^{3-n}(A, D0) \to \mathbb{C}$$

- This pairing can be written as

$$\int \alpha \star \beta$$

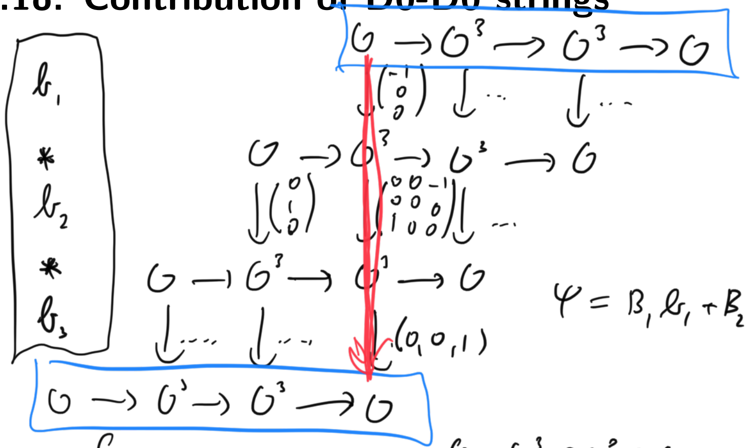
where $\int$ is known as a trace map.

- Let us identify a concrete form of the trace map in our situation.
- First, note that $\mathrm{Hom}^3(D0[1], D0[1])$ is generated by a single element

$$\begin{tikzcd}
 & & & \mathcal{O} \arrow[r] \arrow[d, "1"] & \mathcal{O}^3 \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O} \\
\mathcal{O} \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O}^3 \arrow[r] & \mathcal{O} & & &
\end{tikzcd}$$

- The trace map simply identifies the multiplicative constant with the image in $\mathbb{C}$.
- In the higher-rank situation, we can compose such a map with the standard trace over $X_i$.

## 2.18. Contribution of D0-D0 strings



$$\Psi = B_1 b_1 + B_2 b_2 + B_3 b_3$$

$$\Rightarrow \int dx \quad \begin{matrix} \mathcal{O} \to \mathcal{O}^3 \to \mathcal{O}^3 \to \mathcal{O} \\ \downarrow \\ \mathcal{O} \to \mathcal{O}^3 \to \mathcal{O}^3 \to \mathcal{O} \end{matrix} = -1$$

$$\Rightarrow W = \mathrm{Tr}\, B_1 [B_2, B_3]$$

# 2.19. Contribution of D0-D4 strings

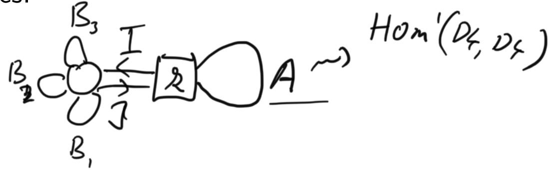
$$\begin{tikzcd}
j & & & & G \arrow[r] \arrow[d] & G^3 \arrow[r] \arrow[d] & G^3 \arrow[r] & G \\
* & & & & G \arrow[r] \arrow[d] & G \arrow[d] & & \\
i & & G \arrow[r] \arrow[dd] & G^3 \arrow[r] \arrow[dd] & G^3 \arrow[r] \arrow[dd] & G & & \\
* & & & & & & & \\
b_3 & G \arrow[r] & G^3 \arrow[r] & G^3 \arrow[r] & G & & &
\end{tikzcd}$$

$\rightsquigarrow$ 1 MULTIPLE OF THE $\mathrm{Hom}^3(D0, D0)$

$$\rightsquigarrow \quad W = \mathrm{Tr}\, B_1 [B_2, B_3] + \mathrm{Tr}\, B_3 I J$$

## 2.20. Turning on Higgs field

- We would like to now comment on how to turn on the expectation value for the Higgs field on non-compact branes.
- Into the quiver, one can obviously include the modes of the non-dynamical fields coming from strings stretched between non-compact branes.




- This is going to lead to a modification of the potential:

$$W = \mathrm{Tr}\, B_1 [B_2, B_3] + J B_3 I + I \underline{A} J$$

- Turning on a constant value for such a Higgs field (that has to be nilpotent to preserve equivariance) leads to a modification of equations of motion.

## 2.21. Flavor symmetries

- Let us now look at $U(1)$ flavor symmetries for which we will introduce the $\Omega$-background.
- Obviously, we can act by $GL(k)$ on the vector space associated to each framing node. Turning on the equivariance for its Cartan subgroup $U(1)^k \subset GL(k)$ plays an important role in understanding the framing by multiple branes and we will briefly comment on this point at the very end of our journey.
- Instead, note that the potential is invariant under

$W = \mathrm{Tr}\, B_1 [B_2, B_3] + J B_3 I$

$\epsilon_1 + \epsilon_2$

$$B_i \to e^{i\epsilon_i} B_i, \quad J \to e^{ia} J$$

for $a$ being a linear combination of $\epsilon_i$ with integral coefficients depending on the choice of the framing brane if we restrict to the subtorus $\boxed{U(1)^2 \subset U(1)^3}$ given by $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$.

- This action on $B_i$ can be traced back to the symmetry of the system that rotates the three coordinate planes in $\mathbb{C}^3$.
- Let us write by $\mathbb{C}_{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}$ for integers $n_i$ the represenation of $U(1)^3$ given by

$$\mathbb{C}_{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3} \to e^{i(n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3)}\mathbb{C}_{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}$$

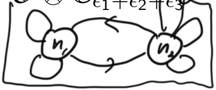
- We can then lift the projective resolution of the D0-brane into the equivariant complex

$$\mathcal{O} \xrightarrow{\begin{pmatrix} -x_1 \\ x_2 \\ -x_3 \end{pmatrix}} \mathcal{O} \times \mathbb{C}_{\epsilon_1} \oplus \mathbb{C}_{\epsilon_2} \oplus \mathbb{C}_{\epsilon_3} \xrightarrow{\begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x & 0 \end{pmatrix}}$$

D4 on $\mathcal{O}(-1) \to \mathbb{P}^1$

$$\mathcal{O} \otimes \mathbb{C}_{\epsilon_2+\epsilon_3} \oplus \mathbb{C}_{\epsilon_1+\epsilon_3} \oplus \mathbb{C}_{\epsilon_1+\epsilon_2} \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} \mathcal{O} \otimes \mathbb{C}_{\epsilon_1+\epsilon_2+\epsilon_3}$$

$\boxed{\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1}$



- Lifting everything into the equivariant map, we see that $B_i$ must transform as $\mathbb{C}_{\epsilon_i} \otimes \mathbb{C}^{n^2}$

$$B_1: \quad \begin{tikzcd} & \boxed{\mathbb{C}_0} \arrow[r] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & \mathbb{C}_{\epsilon_1} \oplus \mathbb{C}_{\epsilon_2} \oplus \mathbb{C}_{\epsilon_3} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \\ \mathbb{C}_0 \arrow[r] & \boxed{\mathbb{C}_{\epsilon_1}} \oplus \mathbb{C}_{\epsilon_2} \oplus \mathbb{C}_{\epsilon_3} \arrow[r] & \cdots \arrow[r] & \cdots & \end{tikzcd}$$

$\rightsquigarrow$ WEIGH $(B_1) = \epsilon_1$

ANALOGOUSLY FOR $B_2, B_3, I, J$

- Remember that the trace map $\int$ was given in terms of a linear map $\mathrm{Hom}^3(D0[1], D0[1]) \to \mathbb{C}$ sending a fixed generator to one. Lifting to an equivariant map, this generator is of weight $e^{i(\epsilon_1+\epsilon_2+\epsilon_3)}$. The condition $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ can be thus traced back to the requirement of the invariance of the trace map.
- Let me also mention a slightly different perspective. If we were to deal with D6-branes, we would identify $\mathrm{Hom}^*(D6, D6) = H^{*,0}_{\bar{\partial}}(\mathcal{O}_X)$ with the trace map being the 6d holomorphic Chern-Simons functional

$$\int_X \alpha \wedge \Omega$$

where $\Omega$ is the Calabi-Yau volume form. In our case, $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ and we can see that its invariance requires $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$.