Random hyperbolic surfaces via flat geometry

(joint w/ James Farre)

1st half: Random hyperbolic surfaces
(mostly independent of James's talk)

2nd half: Applying the conjugacy
(will assume definitions)

Tagline: The conjugacy is a machine for turning theorems in flat geometry into theorems in hyperbolic geometry.

"Focalize through the lens of random hyperbolic surfaces"

What is a random hyperbolic surface?

Hyperbolic surfaces:

Uniformization:

$\{\text{cx str}\} \longleftrightarrow \{\text{hyp str}\}$

Packaged together into Teichmüller/moduli spaces

$$\mathcal{T}_g = \widetilde{\mathcal{M}_g}$$

$$\mathcal{T}_g \times \mathcal{ML}_g / \mathrm{Mod}_g \longleftrightarrow T^*\mathcal{M}_g$$

!!

$$\mathcal{P}\mathcal{M}_g$$

$$(X, \lambda) \longmapsto (X, d\ell_\lambda)$$

P a pants decomposition

Fenchel-Nielsen coords

$$\mathcal{T}_g \cong \mathbb{R}_{>0}^{3g-3} \times \mathbb{R}^{3g-3}$$

lengths (ℓ_i) twists (τ_i)

$$\omega = \sum_i d\ell_i \wedge d\tau_i$$

[Wolpert]

WP symplectic form

"Random" = generic w.r.t. some measure

FN coords $\rightsquigarrow$ WP symplectic form

$\rightsquigarrow$ WP vol. form on $\mathcal{M}_g$ μ_{WP}

Really, want to consider Leb. on $T^1\mathcal{M}_g$.

(just like considering Haar on frame bundle instead of induced measure on symm. space)

$\Rightarrow$ Consider μ_{Mirz} on $\mathcal{P}^1\mathcal{M}_g \cong T^1\mathcal{M}_g \xrightarrow{\pi} \mathcal{M}_g$.

$$\pi_* \mu_{Mirz} = \underbrace{B(X)}_{= \text{volume of } T_X^{\leq 1}\mathcal{M}_g} \mu_{WP}.$$

[Mirzakhani] μ_{Mirz} is ergodic w.r.t. earthquake flow.

(Hamiltonian of hyp. length)

"But still: how do you go about building these?)

<u>Mirzakhani's recipe for building random surfaces</u>

Pick a ~~pants decomp. P~~ multicurve γ

1) Fix random lengths, up to threshold L.

<u>i.e.</u>, pick $(\underline{\ell}) \in \mathbb{R}^{3g-3}$ w/ $\|\underline{\ell}\|_1 \leq L$ (or $= L$)

1.5) Fix random hyp str. on $S \setminus \gamma$, w/ given ∂ lengths

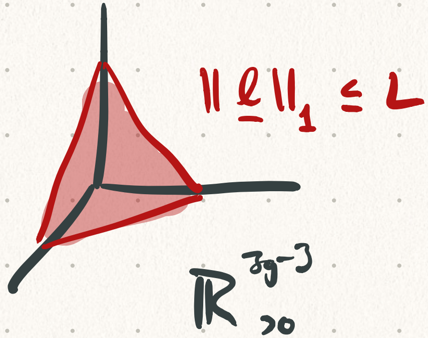
2) Glue w/ random twists $\leftarrow$ torus worth of twisting

$\forall T$, defines a measure $\nu^L_{\cancel{P}\,\gamma} =$ (random lengths) $\times$ (random twists) $\times$ (random complement)

<u>[Mirzakhani]</u>: as $L \to \infty$, $\nu^L_{\cancel{P}\,\gamma} \xrightarrow{*} \mu_{Mirz}$ (on $P^1\mathcal{M}_g$)

[Mirzakhani]:

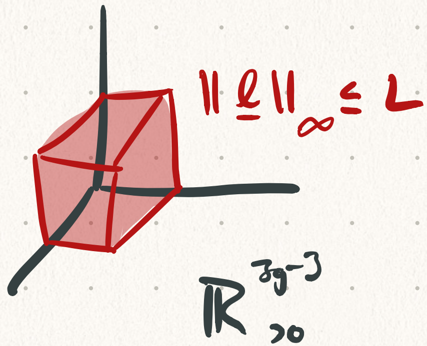
$\mathbb{T}^{3g-3}$ bundle over



[Arana-Herrera, Liu]:

generalize "threshold" to $\|\underline{\ell}\|_p \le L$ (or $=L$) $p \in [1,\infty]$

$\mathbb{T}^{3g-3}$ bundle over



Q: How much of this randomness is really necessary?

(M, AH, L all have $6g-6$ degrees of randomness)

Generalized

Twist torus conjecture: fix $\underline{b} \in \mathbb{R}_{>0}^{3g-3}$

pants decomp

Fix P, set $\underline{\ell} = \cancel{(L, \dots, L)}\ \underline{b} \cdot L$ & take random twists.

$\rightsquigarrow$ immersed torus in $\mathcal{M}_g$ w/ leb. measure $\tau_P(\underline{b}L)$.

$$\text{Do } \tau_P(\underline{b}L) \xrightarrow{*} \mu_{\text{Mirz}} \text{ as } L \to \infty?$$

[Arana-Herrera, 21+]: For a.e. $\underline{b}$,

Still $6g-6$ degrees of randomness!

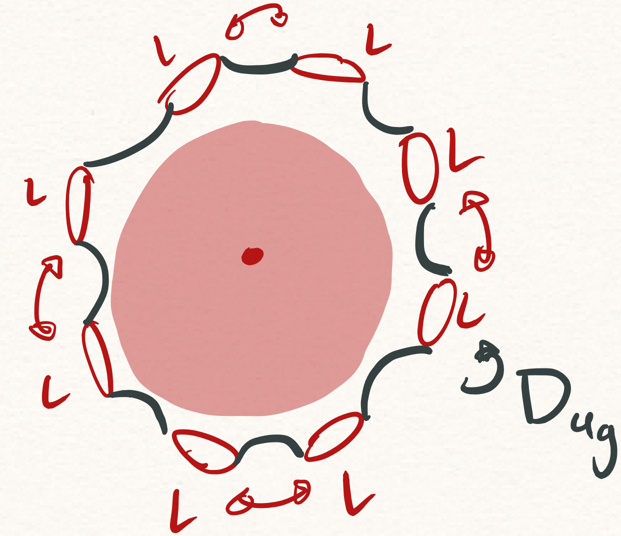
$$\tau_P(\underline{b}L) \xrightarrow{*} \mu_{\text{Mirz}}$$

outside a set of L of lower density 0.

Know NO specific examples!

(in part., no idea for the original example)

Cautionary example: "Regular plumbing fixtures"



$(n_1,\dots,n_k)$-holed spheres D_{an_i}

Take a ~~2g-holed sphere~~ w/ ~~D_{4g}~~ symmetry, then glue together w/ random twists.

$\rightsquigarrow \mathbb{T}^{\cancel{g}\ \sum n_i/2} \subset \mathcal{M}_g$, measure $\tau_{reg}^{\cancel{g}\ (n_1,\dots,n_k)}(L)$

What happens as $L \to \infty$?

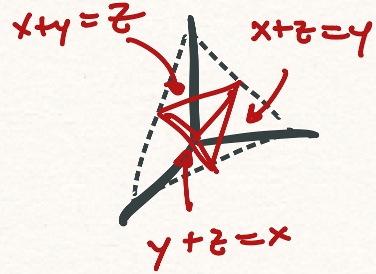
(Do these measures even converge?)

injrad at center is bounded below $\Rightarrow$ can't give mass to

(injrad bdd above)

$\Rightarrow$ Looks qualitatively different than $\mathcal{M}_{Mirz}$!

Theorems [C-Farre, 21+]



Twist tori:

& $\forall \underline{b}$ not in a union of hyperplanes

$\forall$ pants decomp. P, $\forall$ $\tau_P(\underline{b}e^t) \xrightarrow{*} \mu_{Mirz}$

outside a set of t of (upper) density 0.

Regular plumbing fixtures

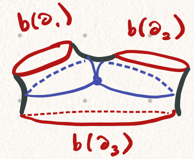
$\tau_{reg}^{2g}(c^t)$ equid. to something singular to μ_{Mirz}

outside a set of t of (upper) density 0.

☆ Show more detailed slide after writing this one.

Theorem [C.–Farre, 21+] "Twist tori from pants decompositions"

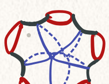
Let P be a pants decomposition of S, $\underline{b} \in \mathbb{R}^P_{>0}$. Set

$$\Delta_P(\underline{b}) := \#\{\text{pants of } P \mid b(\partial_1) + b(\partial_2) = b(\partial_3)\} = \#\Big($$  $$\Big)$$

- If $\Delta_P(\underline{b}) = 0$, then $\tau_P(e^t \cdot \underline{b}) \xrightarrow[t \notin \mathbb{Z}]{*} \mu_{Mirz}$
- If $\Delta_P(\underline{b}) > 0$, then $\tau_P(e^t \cdot \underline{b}) \xrightarrow[t \notin \mathbb{Z}]{*} \mu_{\underline{b}}$ singular w.r.t. μ_{Mirz}, P-ergodic
- If $\Delta_P(\underline{b}) \le 2g-5$, or $\Delta_P(\underline{b}) = 2g-2$ & combinatorial conditions, then $\mu_{\underline{b}}$ is the stratum measure corresp. to $Q^1\mathcal{M}_g(2^{\Delta_P(\underline{b})}, 1^{4g-4-2\Delta_P(\underline{b})})$

Theorem [C.–Farre, 21+] "Regular plumbing fixtures"

Let γ cut S into $(n_1, \dots, n_k)$-holed spheres. Then so long as ≥ 3 of n_i are odd,

 $\tau_{reg}^{(n_1,\dots,n_k)}(e^t) \xrightarrow[t \notin \mathbb{Z}]{*}$ Stratum measure $((n_1+2)^2, \dots, (n_k+2)^2)$

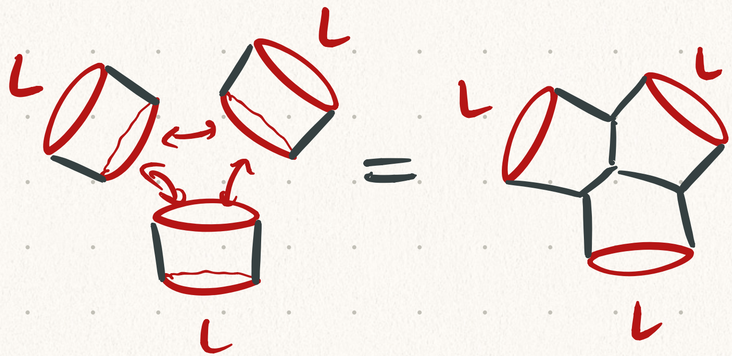
Break

Analogous Q in flat geometry:

Build surfaces out of flat pants of length $= L$

$\rightsquigarrow$ torus $\mathbb{T}^{3g-3} \subset Q^1\mathcal{M}_g$

& measure $\nu_P(L)$



What happens as $L \to \infty$? $\nu_P(L) \xrightarrow{*} ?$

Lots more tools here!

(we'll see an answer later in the talk)

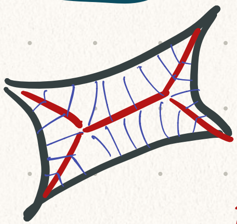
Recap of the conjugacy [C-Farre, 21]

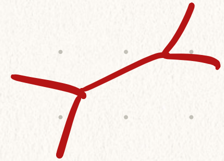
Hyperbolic — Borel measurable bijection — **Flat**

$$\mathcal{O} : \mathcal{P}^1\mathcal{M}_g \rightleftarrows \mathcal{Q}^1\mathcal{M}_g : R$$

μ_{Mirz} μ_{MV}

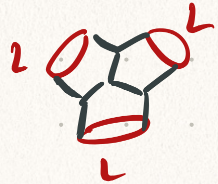
(X, λ) $q(\mathcal{O}_\lambda(X), \lambda)$





Spine of $\mathcal{O}_\lambda(X)$ Horiz. Separatrices

Twist tori (τ_P) Twist tori (ν_P)



Plan: 1) Understand equid. of flat twist tori

2) Pull back along $\mathcal{O}$ (push forward by R)

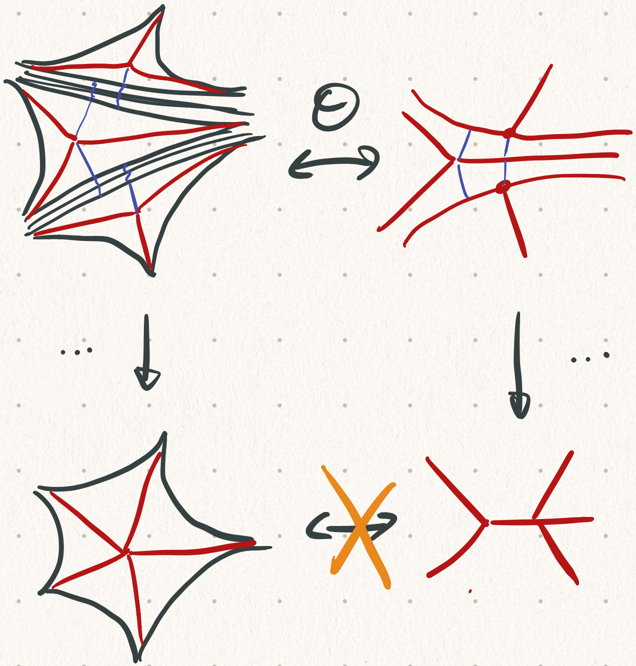
Pulling back equidistribution

"It's obvious you can pull back equid. along homeomorphisms..."

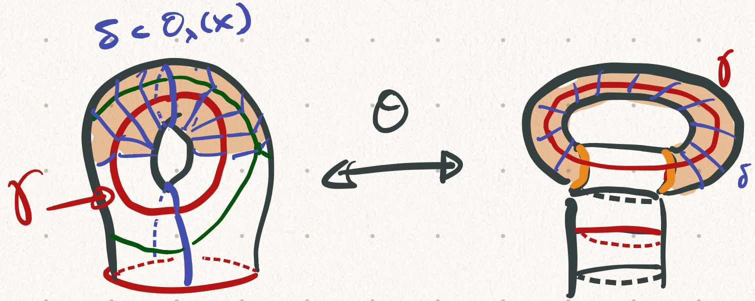
What does it mean to pullback along Θ?

⚠ Θ is neither continuous nor proper.

Discontinuous:



Improper:



As the weight on $\gamma \to 0$,
flat surface $\to \infty$ in $Q'M_g$.

Theorem [C.-Farre, 21+]

(comp. of stratum of $Q^1\mathcal{M}_g$)

Suppose $\nu_n \xrightarrow{*} \nu$ on Q, where ν *affine.*

Then $R_*(\nu_n) \longrightarrow R_*(\nu)$ on $\mathcal{P}^1\mathcal{M}_g$.

Assuming this, let's prove twist torus conjecture.

$= \overline{SL_2\mathbb{R} \cdot (\text{twist torus})}$

1) [Eskin-Mirzakhani-Mohammadi]: $\frac{1}{T}\int_0^T (g_t)_*\nu_1\, dt \xrightarrow{*} \nu_{\mathcal{M}}$

2) [Forni]: $\exists\, Z$ of 0 density s.t. $(g_t)_*\nu_1 \xrightarrow[t \notin Z]{*} \nu_{\mathcal{M}}$

3) [Apisa-Wright]: $\mathcal{M} = Q^1\mathcal{M}_g(1^{4g-4})$.

$\Rightarrow \quad R_*((g_t)_*\nu_1) = \tau_p(e^t) \xrightarrow[t \notin Z]{*} R_*\mu_{MV} = \mu_{Mirz}.$

3) in more detail:

≈ # of horiz cylinders you can twist on

[Apisa–Wright, 21]: Classification of "high rank" orbit closures.

Corollary: If $\mathcal{M} \subset Q'\mathcal{M}_g(\underline{\kappa})$ has full rank & $\underline{\kappa}$ has ≥ 6 odd zeros, $\mathcal{M}$ = stratum.

Numerology:

$$\mathrm{rk}(\text{stratum}) = g + \frac{\#\text{ odd zeros}}{2} - 1 = g + \frac{4g-4}{2} - 1 = 3g-3$$

$$\mathrm{rk}(\mathcal{M}) \geq \#\text{ of cyls} = 3g-3$$

kind of a lie, really computing in orientation cover

$\Rightarrow$ $\mathcal{M} = \overline{SL_2\mathbb{R}\cdot(\text{twist torus})}$ has full rank $\Rightarrow$ stratum.

Idea of the equidistribution theorem

Point: understand continuity properties.

Vanishing support $\Rightarrow$ discont., improper

only way this can happen!

Theorem: [C-Farre, 21+]

$T \times \mathcal{ML} \xrightarrow{\Theta} Q\mathcal{M}_g$ is continuous, & is a homeomorphism on substrata.

"measure + Hausdorff top"

Still, to get thm need to understand cont. of R!

Thin parts

Lemma: R ($= \Theta^{-1}$) is proper.

Pf: Suppose $(X, \lambda) \in \mathcal{P}^1\mathcal{M}_g$, & γ short on X.

$\ell_X(\lambda) = 1 \Rightarrow i(\gamma, \lambda)$ <u>small</u>.

Passing through collar nbhd adds definite length!

$\Rightarrow$ on $\Theta(X, \lambda)$, have

$$\left.\begin{aligned} |\mathrm{Re}|(\mathrm{hol}(\gamma)) &\le \ell_X(\gamma) \\ |\mathrm{Im}|(\mathrm{hol}(\gamma)) &\le i(\gamma, \lambda) \end{aligned}\right\} \text{both small}$$

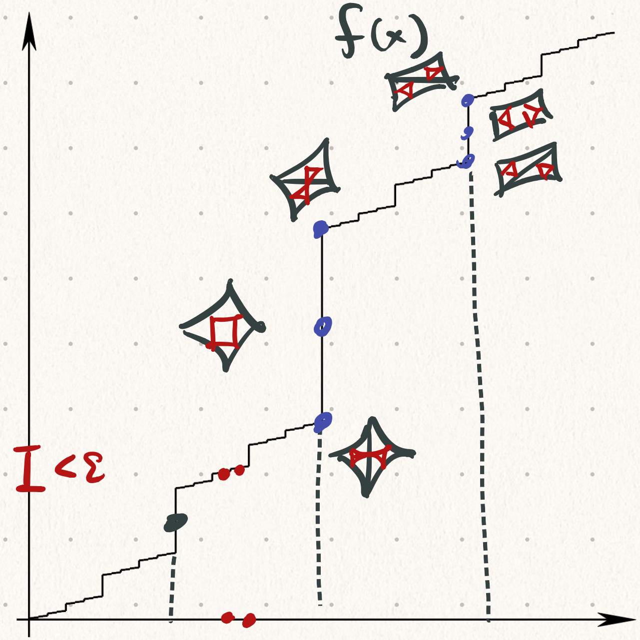
So γ is short on $\Theta(X, \lambda)$, so $\Theta(X, \lambda)$ is thin. $\square$

⚠ <u>Not</u> necessarily true that γ is <u>conformally</u> short

(might be contained in small subsurface)

Continuity of R:

- Discontinuities come from horizontal saddles
- Longer saddle
  $\Updownarrow$
  Smaller discontinuity



Analogy: (inverse) Cantor function

$\forall \varepsilon$, ~~f~~ R is continuous @ scale ε on the thick part outside of a finite set of hyperplanes $(\mathrm{Im}(\mathrm{hol}(s_i)) = 0)$

This is enough to push forward w-* convergence!

Further applications

1) Other equidistributing tori (eg $\mathcal{T}_{reg}(L)$)

2) Pull back Chaika-Khalil-Smillie:

→ Existence of non-generic points for $t\bar{Q}$,

→ Non-closedness of space of EQ ergodic measures

3) Pull back Chaika-Weiss on REL:

→ Equid. of "hyperbolic Schiffer deformations"