Euler numbers of Hilbert schemes of points on simple surface singularities

HIRAKU NAKAJIMA (Kavli IPMU)

June 25 22:15 @ tokyo

—— ZOOM ——

Qolloquium : Quivers, Representations, and Resolutions

§1. Gyenge–Némethi–Szendrői conjecture

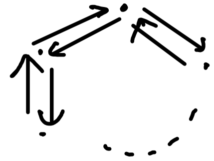
$\Gamma \subset SU(2)$ nontrivial finite subgroup

$\longleftrightarrow$ ADE Dynkin diagram

$\longleftrightarrow$ affine quiver of type ADE

$Q_0 = \{0, 1, \dots, n\}$ vertex

e.g. A_n $\Gamma = \mathbb{Z}/(n+1)\mathbb{Z}$



$\zeta \in \mathbb{Q}^{Q_0}$: stability parameter

$\vec{v}, \vec{w} \in \mathbb{Z}_{\geq 0}^{Q_0}$: dimension vectors

$\mathfrak{M}_\zeta(\vec{v}, \vec{w})$: quiver variety $\left\{ \bigoplus_a \mathrm{Hom}(V_{o(a)}, V_{i(a)}) \oplus \bigoplus \begin{matrix} \mathrm{Hom}(W_i, V_i) \\ \oplus \\ \mathrm{Hom}(V_i, W_i) \end{matrix} \right\} \Big/\!\!\Big/_{\zeta} \prod GL(V_i)$

$\cdot$ moment map $= 0$

We consider only $\vec{w} = \Lambda_0$ (1 at vertex 0, 0 on other vertices) hereafter.

<u>Easy Facts</u> (1) Take $\zeta = 0$.

$$\mathfrak{M}_0(\vec{v}, \Lambda_0) \cong \mathrm{Sym}^N(\mathbb{C}^2/\Gamma) \qquad N = \min(v_i)$$

(2) Take generic ζ ($\Leftrightarrow \mathfrak{M}_\zeta(\vec{v},\vec{w})$ is smooth)

$$\mathfrak{M}_\zeta(\vec{v},\Lambda_0) \underset{\text{diffeo.}}{\simeq} \mathrm{Hilb}^N(\widetilde{\mathbb{C}^2/\Gamma}) \qquad N = \frac{1}{2}\dim \mathfrak{M}_\zeta(\vec{v},\Lambda_0)$$

<u>Th</u> [N'02]

$$\sum_{\vec{v}} \chi(\mathfrak{M}_\zeta(\vec{v},\Lambda_0)) \prod_{i=0}^{n} e^{-v_i\alpha_i} = \prod_{m=1}^{\infty}(1-e^{-m\delta})^{-(n+1)} \sum_{\vec{m}\in\mathbb{Z}^n} e^{-\frac{(\vec{m},C\vec{m})}{2}\delta} \prod_{i\neq 0} e^{-m_i\alpha_i}$$

Cartan matrix

$$\overset{!!}{Z}$$

where δ = primitive positive imaginary ~~root~~ $= \underset{1}{\overset{\shortparallel}{a_0}}\alpha_0 + \cdots + a_n\alpha_n$
$= \alpha_0 + a_1\alpha_1 + \cdots + a_n\alpha_n$

<u>Remark.</u> RHS = character of the basic representation of $\mathfrak{g}_{\mathrm{aff}}$
$\times$ char. Fock rep. of Heisenberg algebra

<u>Conjecture</u> [GNS]

$h^\vee$ = dual Coxeter number (e.g. n for A_{n-1})

$$\sum_{m=0}^{\infty} \chi(\mathrm{Hilb}^m(\mathbb{C}^2/\Gamma))\, e^{-m\delta} = Z \Big|_{e^{-\alpha_i} = \exp\left(\frac{2\pi\sqrt{-1}}{h^\vee+1}\right)\ (i\neq 0)}$$

ideals in $\mathbb{C}[x,y]^\Gamma$

- [GNS] checked the conjecture for type A, D

Theorem [N, 20] Conjecture is true for all ADE.

Remark Proof is case by case (at least for the crucial part)

§2. Reduction to Euler numbers of quiver varieties of finite type

The proof uses the following results

(1) **Th** [Craw-Gammelgaard-Gyenge-Szendrői 19]
Choose stability parameter $\zeta^\bullet$ s.t. $\begin{cases} \zeta^\bullet_i = 0 & (i \neq 0) \\ \zeta^\bullet_0 < 0 \end{cases}$

$$\Longrightarrow \quad \mathfrak{M}_{\zeta^\bullet}(m\delta, \Lambda_0) \underset{\text{isom.}}{\cong} \mathrm{Hilb}^m(\mathbb{C}^2/\Gamma)_{\mathrm{red}}$$

(2) [N '09] Take ζ s.t. $\zeta_i < 0 \quad \forall i$ (generic)

$$\Longrightarrow \quad \mathfrak{M}_\zeta(\vec{v}, \Lambda_0) \xrightarrow{\pi_{\zeta^\bullet,\zeta}} \mathfrak{M}_{\zeta^\bullet}(\vec{v}, \Lambda_0) \quad \text{projective morphism}$$

$$\longrightarrow \mathfrak{M}_0(\vec{v}, \vec{w})$$

Moreover $\mathfrak{M}_{\zeta^\bullet}(\vec{v}, \Lambda_0) = \exists \bigsqcup_{\vec{v}'} \mathfrak{M}^s_{\zeta^\bullet}(\vec{v}', \Lambda_0)$ stratification

and $\pi_{\zeta^\bullet,\zeta}$ is a fiber bundle on each stratum
with fiber $= \mathfrak{L}_\zeta(\vec{v}^s, \vec{w}^s)$: lagrangian subvariety in finite ADE type quiver variety

Here $\begin{cases} \vec{v}^s = \vec{v} - \vec{v}' & (v_0^s = 0 \text{ always}) \\ w^s_i = w_i - (C\vec{v})_i & (i \neq 0) \end{cases}$

One can also show $\vec{v}' \in m\delta - \sum \mathbb{Z}_{\geq 0} \alpha_i$.

$\therefore$ $\underset{\text{stratum}}{\mathfrak{M}^s_{\zeta^\bullet}(\vec{v}', \Lambda_0) \subset} \mathfrak{M}_{\zeta^\bullet}(m\delta, \Lambda_0) \subset \mathfrak{M}_{\zeta^\bullet}(\vec{v}, \vec{w})$ i.e. strata are strata of $\mathrm{Hilb}^m(\mathbb{C}^2/\Gamma)$

Therefore $\chi(\mathrm{Hilb}^m(\mathbb{C}^2/\Gamma))$ is given in terms of Ξ and

Euler numbers of $\mathfrak{L}_\zeta(\vec{v}^s, \vec{w}^s)$ for all $\vec{v}^s, \vec{w}^s$

Difficulty : Although there are several known algorithms to compute $\chi(\mathfrak{L}_\zeta(\vec{v}^s, \vec{w}^s))$, very difficult to manipulate ....

§ 3. Miraculous cancellation

In fact, the above analysis gives much more than $\chi(\mathrm{Hilb}^m(\mathbb{C}^2/\Gamma))$. It gives $\chi(\mathfrak{M}^s_{\zeta\bullet}(\vec{v}', \vec{w}'))$ for all $\vec{v}'$ (all strata).

$\rightsquigarrow$ We should compute specific sum of Euler numbers of $\mathfrak{L}_\zeta(\vec{v}^s, \vec{w}^s)$

<u>Lemma</u> GNS conjecture follows from

$$☆ \quad \sum_{\vec{v}^s} \chi(\mathfrak{L}_\zeta(\vec{v}^s, \vec{w}^s))\, e^{\sum_{i=1}^{n} w_i^s \Lambda_i - v_i^s \alpha_i} \Big|_{e^{-\alpha_i} = \exp(\frac{2\pi\sqrt{-1}}{h^\vee+1})} = 1$$

<u>Remark</u> $\bigoplus_{\vec{v}^s} H_*(\mathfrak{L}_\zeta(\vec{v}^s, \vec{w}^s))$ is a representation of $\mathfrak{g}_{\mathrm{fin}}$ (In fact $U_q(\hat{\mathfrak{g}})|_{q=1}$) called a standard module.

$\therefore$ LHS = specialization of character of a standard module
(called quantum dimension
(appeared in repr. of quantum group at roots of 1.

$\zeta = \exp(\frac{2\pi i}{2(h^\vee+1)})$ in this case

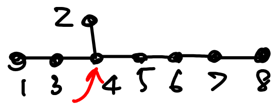
Th. ☆ is true.

(proof) enough to show ☆ for $\vec{w}^s = \Lambda_i$ ($\exists i \neq 0$) by the torus action

This case Euler numbers are known.

e.g. A_n $\rightsquigarrow$ all quiver varieties are pts (or ϕ)

E_8 $\rightsquigarrow$ computed by supercomputer

(it took 350 hours for 

)

$\rightsquigarrow$ ☆ is true.

In fact, ☆ is the simplest case of a conjecture by Kuniba '93
(see also Kuniba - Nakanishi - Suzuki 11)

$\rightsquigarrow$ I will explain it on July 2 at LAGOON.

Good Night and See you!